Modeling Flight Over a Spherical Earth

David Zitter

General Dynamical Equation

$$\mathbf{T} + \mathbf{A} + m\mathbf{g} = m\left[\frac{d\mathbf{V}}{dt} + 2\,\omega_e \times \mathbf{V}\right]$$

- T – thrust
- A – aerodynamic force
- m – mass
- ω_e – angular velocity of the Earth
- V – velocity of the aircraft with respect to the Earth

Coordinate Systems

- Earth axes system
- Curvilinear ground system
  - X-coordinate measured along the fundamental parallel
  - Y-coordinate measured along the fundamental meridian
  - Z-coordinate measured radially
- Local horizon system
- Body axes system

Angular Relationships

- Local horizon-Earth axes

$$\begin{bmatrix} i_h \\ j_h \\ k_h \end{bmatrix} = \begin{bmatrix} \cos(X/r_o) & 0 & \sin(X/r_o) \\ -\sin(X/r_o)\sin(Y/r_o) & \cos(Y/r_o) & \cos(X/r_o)\sin(X/r_o) \\ -\sin(X/r_o)\cos(Y/r_o) & -\sin(Y/r_o) & \cos(X/r_o)\cos(Y/r_o) \end{bmatrix} \begin{bmatrix} i_e \\ j_e \\ k_e \end{bmatrix}$$

$$X = r_o\tau \quad Y = r_o\lambda \quad \tau = \text{longitude} \quad \lambda = \text{latitude}$$

- Wind axes-local horizon

$$\begin{bmatrix} i_w \\ j_w \\ k_w \end{bmatrix} = \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ -\sin\chi & \cos\chi & 0 \\ \sin\gamma\cos\chi & \sin\gamma\sin\chi & \cos\gamma \end{bmatrix} \begin{bmatrix} i_h \\ j_h \\ k_h \end{bmatrix}$$

$$\gamma = \text{velocity pitch angle} \quad \chi = \text{velocity yaw angle}$$

Kinematic Relationships

- $\mathbf{V} = \dfrac{d\mathbf{EO}}{dt}$ where EO is the vector joining point E on the surface of the Earth with the aircraft, point O
- Collinear with the x_b-axis, $\boldsymbol{V} = V\mathbf{i}_w =$

$$V(\cos\gamma\cos\chi\,\mathbf{i_h} + \cos\gamma\sin\chi\,\mathbf{j_h} - \sin\gamma\mathbf{k_h})$$

$$\dot{X} = V\frac{r_o}{r_o + h}\frac{\cos\gamma\cos\chi}{\cos(Y/r_o)} \qquad \dot{Y} = V\frac{r_o}{r_o + h}\cos\gamma\cos\chi$$

$$\dot{h} = V\sin\gamma$$

Dynamic Relationships

$$\mathbf{T} + \mathbf{A} + m\mathbf{g} = m\left[\frac{d\mathbf{V}}{dt} + 2\,\boldsymbol{\omega_e} \times \mathbf{V}\right]$$

$$\boldsymbol{T} = T[\cos\epsilon\,\mathbf{i}_w - \sin\epsilon\,\mathbf{k}_w] \qquad \boldsymbol{A} = -(D\mathbf{i}_w + \mathrm{Q}\mathbf{j}_w + \mathrm{L}\mathbf{k}_w)$$

$$\boldsymbol{g} = g[-\sin\gamma\,\mathbf{i}_w + \cos\gamma\,\mathbf{k}_w] \qquad \frac{d\mathbf{V}}{dt} = \dot{V}\mathbf{i}_w + \mathrm{Vr}_w\mathbf{j}_w - \mathrm{Vq}_w\mathbf{k}_w$$

$$g = g_o\left(\frac{r_o}{r_o + h}\right)^2 \qquad a_c = 2\omega_e \times V = 2V(\mathrm{r}_{ew}\mathbf{j}_w - \mathrm{q}_{ew}\mathbf{k}_w)$$

ϵ = thrust angle of attack

System of Differential Equations

$$\dot{X} = V \frac{r_o}{r_o + h} \cos\gamma$$

$$\dot{h} = V \sin\gamma$$

$$T \cos\epsilon - D - m g_o \left(\frac{r_o}{r_o + h}\right)^2 \sin\gamma = m\dot{V}$$

$$T \sin\epsilon + L - m g_o \left(\frac{r_o}{r_o + h}\right)^2 \cos\gamma = mV[\dot{\gamma} - \frac{V \cos\gamma}{r_o + h} + 2q_{ew}]$$