Draft Due: Thursday, March 1, by 5 p.m.
Final Due: Thursday, March 8, by 5 p.m.

Attached to this assignment are seven design and optimization problems taken from *Applied Numerical Methods for Engineers* by Terrence Akai. Your assignment is to pick one the design problems and solve it to the best of your ability. You may use any software platform that is most convenient for you, but please stick to software that I can run so that I can test your code.

Please write a description of the design problem, your solution technique and your findings. Your write-up should be composed on a computer and should be written so that anyone else in this class can understand it without prior knowledge of the design problem that you've chosen. Please submit your paper and code electronically to `dyong@hmc.edu`.

10.4.1 Problem 1: Design of an Airship Gas Envelope

Airships, also known as ***blimps*** or ***dirigibles***, are lighter-than-air vehicles. The primary mechanism for lift is buoyancy, which is obtained by filling a gas envelope with helium. Airships are used mainly for advertising and for added television coverage of sporting events. Other possible uses are as platforms for border surveillance, for damage assessment following natural disasters, and for early warning and detection systems.

The common shape for the gas envelope of an airship is similar to an ellipsoid. We shall consider a body of revolution whose radius r varies with the coordinate z on the axis of revolution according to

[P1.1] $$r^2 = b^2(1 - 4z^2)(0.5 - z)^c; \qquad -0.5 \leq z \leq 0.5$$

The quantities r and z are lengths that are nondimensionalized with respect to the axial length L of the envelope. The quantity b is a fineness parameter that is related to the maximum radius r_{max} of the envelope. The quantity c is a shape parameter that is related to the location z_m at which the radius is maximum. When c is equal to zero, the envelope is an ellipsoid with z_m equal to zero and r_{max} equal to b. Positive values of z_m produce negative values of c.

The shape of the envelope is to be specified *via* the maximum nondimensional radius r_{max} and its nondimensional location z_m. The parameter c is obtained from z_m by setting the derivative $d[r^2]/dz$ to zero when z is equal to z_m, and is then used to compute the parameter b from Eq. [P1.1]. An envelope is illustrated in Fig. P1–1.

The envelope is to have a volume of 2000 m^3. The nondimensional volume V is found from

[P1.2] $$V = \pi \int_{-0.5}^{0.5} r^2 \, dz$$

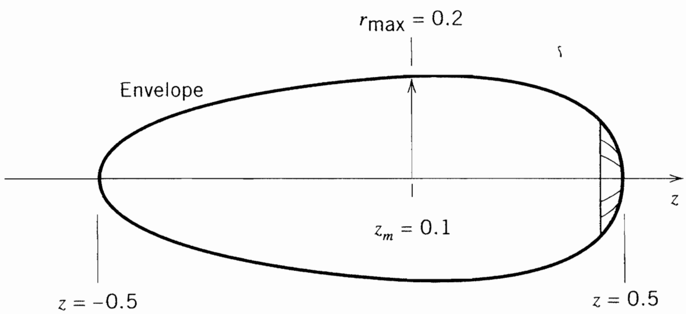

Figure P1–1 Gas envelope with (r_{max} = 0.2) and (z_m = 0.1).

The integral in Eq. [P1.2] may be obtained analytically by performing two integrations by parts. The length L of the envelope (in meters) is then given by

[P1.3] $$L^3 = 2000/V$$

The nondimensional surface area S of the envelope is found from

[P1.4] $$S = 2\pi \int_{-0.5}^{0.5} r\sqrt{1 + (dr/dz)^2}\, dz$$

This integral is best evaluated by one of the numerical integration methods of Chapter 6. Problems 18 and 19 of Chapter 6 are especially useful. It is suggested that the transformation

$$z = -0.5 \cos\theta$$

be used to change the variable of integration to θ between the limits 0 and π. This change clusters the control points for the integration at locations where the integrand varies rapidly. Regardless of the variable of integration, pay special attention to the integrand values at the limits of the integration.

The "cost" of the design is proportional to the actual surface area (SL^2). We wish to maximize the buoyancy, which is proportional to the actual volume (VL^3). We also wish to minimize the drag on the airship. This drag is proportional to ($C_d[r_{max}L]^2$), where C_d is a nondimensional drag coefficient. An empirical relation for the behavior of C_d is

[P1.5] $$C_d = 0.1136 - f\{0.04858 - f(0.01170 - 0.0008167f)\}; \qquad 2 \le f \le 5$$

in which f is the envelope's fineness ratio

[P1.6] $$f = 1/(2r_{max})$$

An appropriate criterion function for the design of the gas envelope is one that embodies the objectives of maximizing buoyancy and minimizing cost and drag. We therefore specify a criterion function ϕ by

[P1.7] $$\phi = LC_d(r_{max})^2 S/V$$

One of the constraints for the problem is that the fineness ratio f in Eq. [P1.6] must be in the range specified in the empirical relation of Eq. [P1.5]. This constraint is imposed for the practical reason that the drag coefficient is available only for this range. The equivalent constraint in terms of r_{max} is

[P1.8] $$0.1 \le r_{max} \le 0.25$$

A second, *ad hoc* constraint is imposed for esthetic as well as physical reasons. The drag coefficient given by Eq. [P1.5] is for conventional "fuselage-like" shapes. To avoid excessive bluntness at the "nose" of the gas envelope, the axial location z_m of the maximum radius is restricted to the range

[P1.9] $$2r_{\max} - 0.5 \leq z_m \leq 0.5 - 2r_{\max}$$

Note that the constraint on z_m allows either end ($z = -0.5$) or ($z = 0.5$) to be the nose of the envelope, and allows only a zero value of z_m when $r_{\max}$ is at the upper limit of the range in Eq. [P1.8].

In summary, the problem is to design the gas envelope of an airship according to the shape given in Eq. [P1.1]. The actual volume of the envelope is to be 2000 m^3, the criterion function ϕ of Eq. [P1.7] is to be minimized, and the constraints of Eq. [P1.8] and [P1.9] are to be satisfied. Values to be reported for the final design are the nondimensional quantities $r_{\max}$ and z_m, the criterion function ϕ, the fineness and shape parameters b and c, the nondimensional volume V and surface area S, the length L of the envelope (in meters), the fineness ratio f, the drag coefficient C_d, and a table of nondimensional r and z. An illustration of the envelope, similar to the one in Fig. P1–1, is also to be provided.

The following properties of an ellipsoid with a unit major axis and a maximum radius $r_{\max}$ may be useful as a preliminary check of the results for V and S. The corresponding values for an ellipsoid (corresponding to a zero value of z_m) are

$$V_e = 2\pi(r_{\max})^2/3$$

$$S_e = \pi(r_{\max}/\beta)\left[\sqrt{\beta^2 - 0.25} + 2\beta^2 \sin^{-1}(0.5/\beta)\right]; \qquad 1/\beta = 2\sqrt{1 - (2r_{\max})^2}$$

An open-ended version of the problem is to design the envelope without the shape function of Eq. [P1.1]. The envelope is still required to be a body of revolution with infinite slopes (dr/dz) at the nose and tail, and one location z_m at which (dr/dz) is zero. The criterion function of Eq. [P1.7] and the constraints on $r_{\max}$ and z_m in Eq. [P1.8] and Eq. [P1.9] remain in effect.

A new constraint to define acceptable body shapes must now be introduced to replace the one that was inherent in the shape function of Eq. [P1.1]. This constraint requires the envelope to be nonconcave and is given by

[P1.10] $$d^2r/dz^2 \leq 0; \qquad -0.5 \leq z \leq 0.5$$

10.4.2 Problem 2: Design of a Plane Truss

Trusses are lightweight structures for supporting heavy loads. Examples of trusses are seen on bridges and on roof structures. A discussion of plane trusses is provided in §2.2.5 and Appendix D.

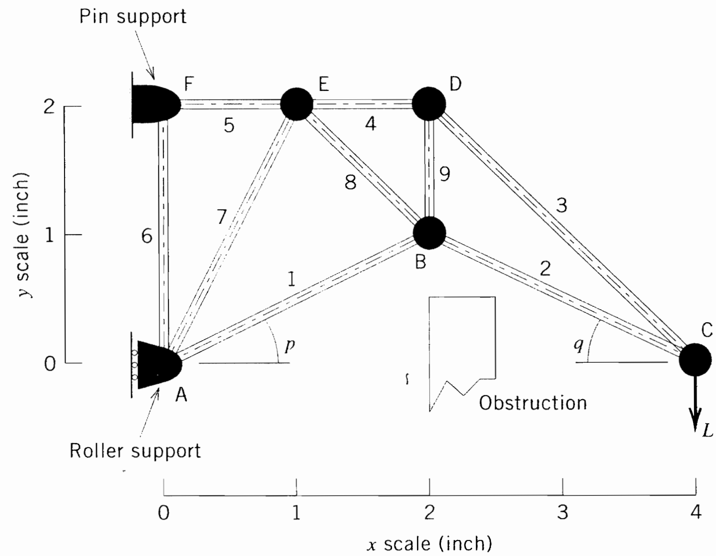

Figure P2–1 Illustration of the truss.

The truss for our design problem consists of six joints and nine members as shown in Fig. P2–1. The joints are labeled A through F, and members are numbered 1 through 9. All lengths are in inches, and the coordinates (x, y) of all joints except B are fixed as follows.

[P2.1]
$$(x_a, y_a) = (0, 0); \qquad (x_c, y_c) = (4, 0);$$
$$(x_d, y_d) = (2, 2); \qquad (x_e, y_e) = (1, 2); \quad (x_f, y_f) = (0, 2)$$

The external load L is in the negative y direction at joint C. The truss has a pin support at joint F; the external force at that joint thus has two components. The support at joint A is a roller support; the external force at that joint has only an x component.

The location of joint B must be such that members 1 and 2 clear an obstruction. Clearance is satisfied by the following conditions on the angles p and q.

[P2.2]
$$p \geq 16°; \qquad q \geq 21°$$

In addition, joint B is constrained to lie in the region formed by members 3, 4, and 7, and it is to be no closer than 0.5 in. from the centerline of any of those members. The objective of the design is to maximize the load per weight of the truss. If $L_{\max}$ is the maximum load that the truss can support without failure, and W is the weight of the truss, the criterion function to be maximized is

[P2.3]
$$\phi = L_{\max}/W$$

The truss is to be constructed of 1/16-in.-diameter steel rods. Let r_m be the length of member m; then the weight W in pounds is given by

[P2.4]
$$W = 0.00087 \sum_{m=1}^{9} r_m$$

The lengths r_m may be obtained by the distance formula (based on Pythagoras' theorem) for a line between two known points.

The determination of the maximum load L_{max} proceeds as follows. We first compute the member forces f_m for a load L equal to 1 lb, with tensions as positive forces and compressions as negative forces in the formulation of the problem. We then compute for each member the fraction Q_m of the maximum safe load that each force f_m represents. The values of Q_m for the nine members are obtained from

[P2.5]
$$Q_m = \begin{cases} f_m/250; \ f_m \geq 0 \\ -(f_m/450)(r_m)^2; \quad f_m < 0 \end{cases}$$

The value for positive f_m is for failure in tension, and the value for negative f_m is for failure in compression due to buckling. The maximum external load L_{max} (in pounds) can now be computed from

[P2.6]
$$L_{max} = 1/\{\max(Q_m)\}$$

An open-ended version of this problem retains the locations of joints A, C, and F. The constraints of Eq. [P2.2] are also retained. The new design problem is to locate the remaining joints B, D, and E so as to maximize (L_{max}/W). The basic truss geometry is to be preserved, and no length r_m is to be less than 0.5 inch.

10.4.3 Problem 3: Design of a Four-Bar Linkage

A ***four-bar linkage*** is shown in Fig. P3–1. It consists of an input crank AB of length p, an output crank OC of length q, and a connecting bar BC of length r. The fourth

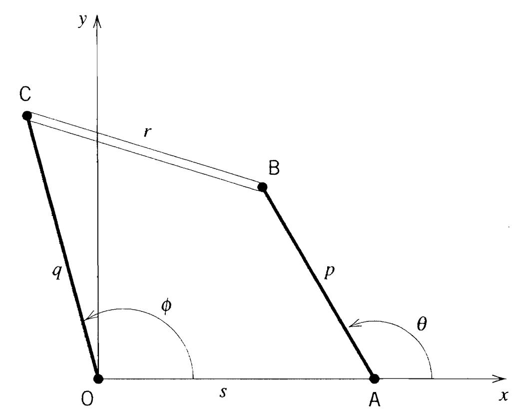

Figure P3–1 Illustration of a four-bar linkage.

bar of length s does not actually exist; rather it is the fixed foundation for the pivot points O and A.

Four-bar linkages are typically used to convert the rotary motion of the input crank to an oscillatory motion of the output crank. The output crank may then drive mechanisms such as windshield wipers, agitators for washing machines, and the oscillating arm of lawn sprinklers.

The pivot O of the output crank is the origin of an (x, y) coordinate system as shown in Fig. P3–1. The angles θ and ϕ of the input and output cranks are measured counterclockwise from the positive x axis.

The maximum output angle $\phi_{\max}$ occurs when θ is equal to θ^+, and the minimum output angle $\phi_{\min}$ occurs when θ is equal to θ^-. The output angles are constrained to lie in the range

$$0 \leq \phi_{\min} < \phi_{\max} \leq \pi \text{ radian} \qquad \text{[P3.1]}$$

The angles $\phi_{\max}$, θ^+, $\phi_{\min}$, and θ^- may be found with the help of Fig. P3–2. Maximum output occurs when AB and BC are fully extended to form a straight line; minimum output occurs when AB and BC overlap.

The angles for minimum and maximum output may be obtained by applying the cosine law to the triangle OAC in each of the configurations. The more general problem of computing ϕ for an arbitrary value of θ is solved with the aid of Fig. P3–3. Two cases are shown — one with point B above the x axis, and one with point B below the x axis.

Using polar coordinates, we obtain the coordinates (x_b, y_b) of point B as

$$x_b = p \cos \theta + s; \qquad y_b = p \sin \theta \qquad \text{[P3.2]}$$

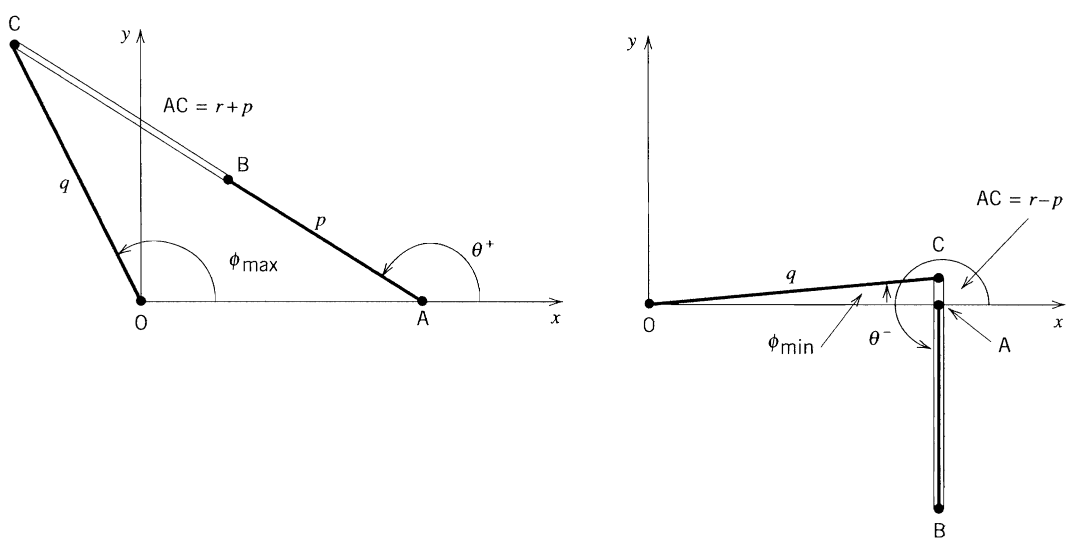

Figure P3–2 Geometries for minimum and maximum output angles.

The distance h from O to B is then found from the distance formula. For either configuration, the cosine law may be applied to triangle OBC to obtain the angle γ. The angle ϕ is then found from

$$\phi = \gamma + \tan^{-1}(y_b/x_b) \qquad \text{[P3.3]}$$

The objective of the design is to choose the lengths p, q, and r with s equal to 1 so that the behavior of the output angle ϕ with θ is as close to sinusoidal as possible. To define sinusoidal for a particular set of lengths, we use the concept of least squares from Chapter 4.

Let ϕ_i be the output angles at input angles

$$\theta_i = i(\pi/12); \qquad i = 0, 1, \ldots, 23 \qquad \text{[P3.4]}$$

The sinusoidal model for ϕ is expressed by

$$\phi_{\text{model}} = \alpha + A \sin(\theta - \beta) \qquad \text{[P3.5]}$$

where

$$\alpha = (\phi_{\text{max}} + \phi_{\text{min}})/2; \qquad A = (\phi_{\text{max}} - \phi_{\text{min}})/2 \qquad \text{[P3.6]}$$

The phase angle β is chosen to minimize the root mean square deviation d_{rms}, which is given by

$$(d_{\text{rms}})^2 = (1/24)\left[\sum_{i=0}^{23}(\phi_i - [\phi_{\text{model}}]_i)^2\right] \qquad \text{[P3.7]}$$

An approximate starting value for β in the minimization of d_{rms} is $[(\theta^+ + \theta^-)/2 - \pi]$.

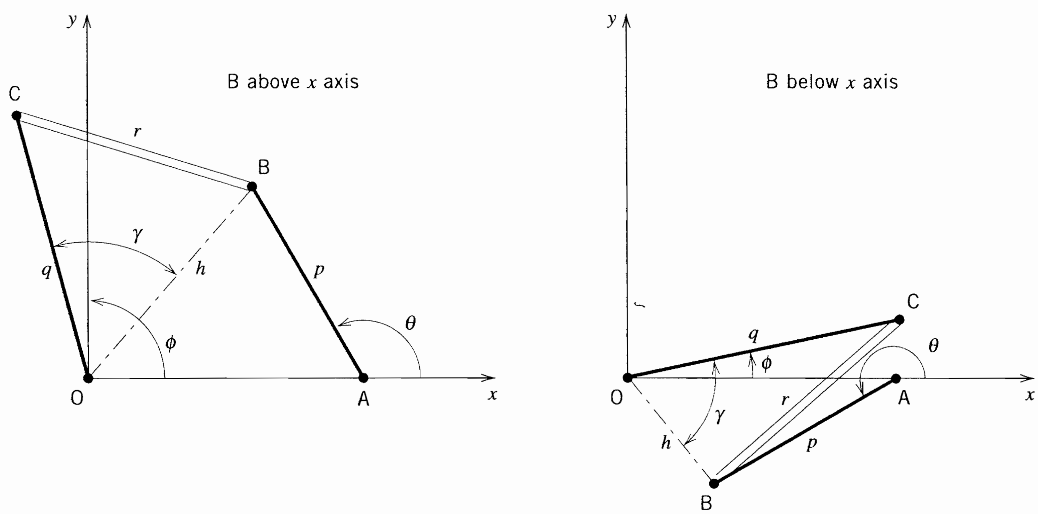

Figure P3–3 Configurations for arbitrary θ values.

The root mean square deviation d_{rms} is the criterion function for the problem. We have already given a constraint in Eq. [P3.1], and we have specified the length s of the fixed bar to be equal to 1. We now give additional constraints for the problem.

The additional constraints include a required range for the amplitude A; namely,

[P3.8] $$0.95 \text{ radian} \leq A \leq 1.00 \text{ radian}$$

Other constraints involve the lengths p, q, and r as follows.

[P3.9] $$0.2 < p < 1$$

[P3.10] $$p < q < 2$$

[P3.11] $$0.2 < r < 2$$

These last three constraints are minimal constraints, which help to size the problem. They may need to be stronger to meet the other constraints of the problem and to satisfy the condition that the link must be geometrically possible for all input crank angles θ.

To determine if the linkage is possible, consider the procedure for computing the output angle ϕ. This procedure involves the use of the cosine law to find the angle γ; that is, γ is the inverse cosine of some argument involving $p, q, r, s,$ and θ. Since the argument of the inverse cosine cannot exceed 1 in magnitude, the link is possible if the minimum and maximum values of the function are at most 1 in magnitude.

10.4.4 Problem 4: Design of a Rack and Pinion Steering Linkage

Steering of a car is usually accomplished by turning the front wheels. If the wheels have the same steering angle, the circular arcs that the individual wheels try to follow have different centers. Although the centers are different, the car will seek to turn about a common center. The tires will therefore have to undergo some deformation or sliding (also known as scrubbing) to accommodate the motion. Both of these are undesirable because they adversely affect the lateral handling of the car, and they impose severe stresses on the materials and mechanisms of the wheel assemblies.

To avoid deformation and scrubbing, all four wheels should roll in circular paths about a common center of rotation. To achieve this, the inner wheel should be turned through a greater angle than the outer wheel. The ***ideal*** steering angle $(\theta_i)_{\text{ideal}}$ of the inner wheel for a given steering angle θ_o of the outer wheel is illustrated in Fig. P4–1.

The steering angles are related through the right-angled triangles CAA′ and CBB′ in Fig. P4–1. The wheels pivot about points A and B, each of which is at a distance d from the centerline of the car. L is the car's wheelbase and represents a common height for the two triangles; the bases are b_i for CAA′ and b_o for CBB′.

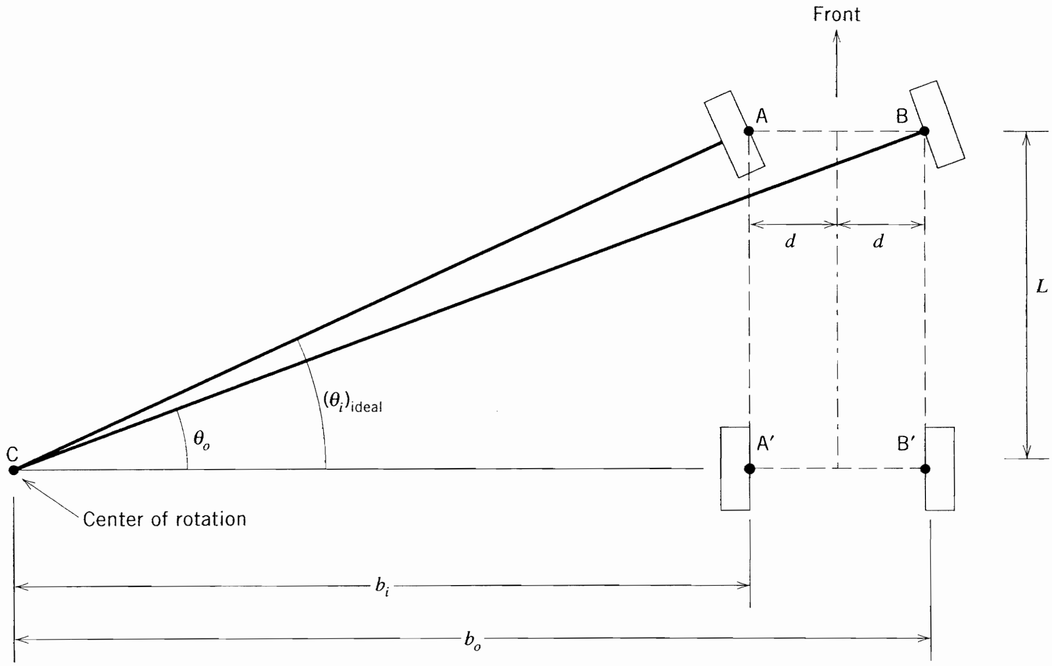

Figure P4–1 Front-wheel steering of a car.

Simple geometry gives us

$$\tan\,\theta_o = L/b_o;\quad \tan(\theta_i)_{\text{ideal}} = L/b_i;\quad b_i = b_o - 2d$$

L is given as 2.40 m, and d is given as 0.65 m. When θ_o is also specified, we obtain

[P4.1] $$\tan(\theta_i)_{\text{ideal}} = L/[(L/\tan\theta_o) - 2d]$$

We shall use Eq. [P4.1] to help us design a ***rack and pinion*** steering system, which is shown in Fig. P4–2 and is described as follows.

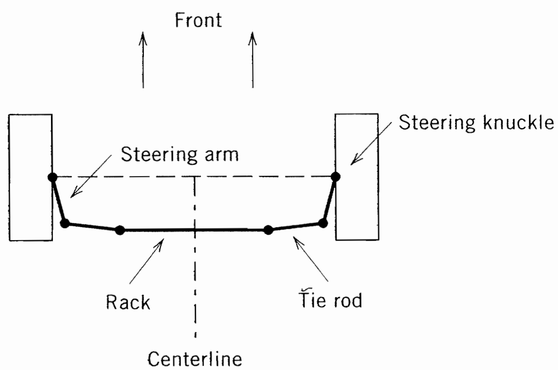

Figure P4–2 Illustration of a rack and pinion linkage.

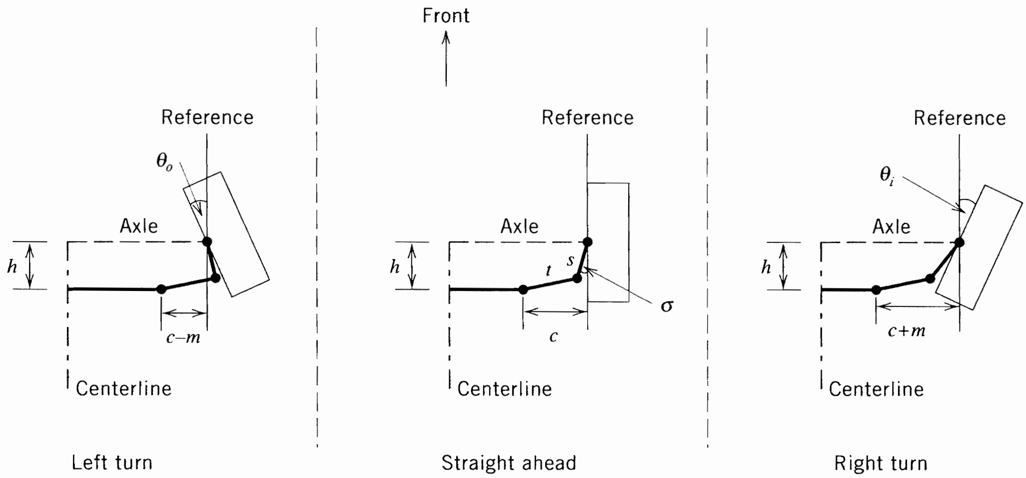

Figure P4–3 Operation of the rack and pinion linkage.

A toothed rack is driven by a small gear known as a ***pinion***. The pinion converts the rotary steering wheel input to linear motion of the rack. The operation of the linkage is illustrated in Fig. P4-3.

The rack is at a fixed distance h from the axle line of the front wheels. When the rack is in a neutral position, the end joint is at a distance c from a reference line through the steering knuckle. A rack motion m produces ***actual*** steering angles θ_o and θ_i (depending on the direction) by pivoting the entire wheel and steering arm assembly *via* tie rods of length t; the tie rods are necessary because the circular paths of the arms and the linear path of the rack are not directly compatible. Each steering arm has a length s and an inboard angle σ.

The lengths c and h are fixed at 0.30 m and 0.22 m, respectively, for this design. We now consider how to obtain the length t of the tie rod for given inputs of s and σ, and how to obtain the required motion m and the inner steering angle θ_i for a given value of θ_o. Let us consider an (x, y) coordinate system with origin at the right steering knuckle, x to the right and coincident with the axle line, and y to the front and coincident with the reference line.

We see from the straight-ahead geometry that the ends of the tie rod are at (x, y) equal to $(-c, -h)$ and at (x, y) equal to $(-s \sin\sigma, -s \cos\sigma)$. We therefore obtain t from

[P4.2] $$t^2 = (c - s\sin\sigma)^2 + (h - c\cos\sigma)^2$$

The rack motion m required to produce a steering angle θ_o in the left-turn geometry requires more manipulation. We can show that m is related to θ_o by

[P4.3] $$m = c - s\sin(\sigma - \theta_o) - \sqrt{t^2 - [h - s\cos(\sigma - \theta_o)]^2}$$

The steering angle θ_i produced by the same rack motion is found from the right-turn geometry. It is given by

[P4.4] $$\alpha + \sigma + \theta_i = \sin^{-1}\{([c + m]^2 + h^2 + s^2 - t^2)/(2s\,p)\};$$
$$p^2 = h^2 + (c + m)^2; \qquad \tan\,\alpha = h/(c + m)$$

The objective of the design is to specify s and σ so that the root-mean-square deviation of θ_i from $(\theta_i)_{\text{ideal}}$ is minimized for θ_o ranging from 0 to 30°. The root mean square deviation d_{rms} is the criterion function and is given by

[P4.5] $$(d_{\text{rms}})^2 = (1/30)\int_0^{30} [\theta_i - (\theta_i)_{\text{ideal}}]^2\, d\theta_o$$

Constraints on the geometry are imposed as follows. The length s of the steering arm is required to satisfy

[P4.6] $$s \le 0.22 \text{ m}$$

To make sure that the joints clear the wheel, the angle σ must satisfy

[P4.7] $$s \sin\sigma \ge 0.02 \text{ m}$$

In Eq. [P4.6] and [P4.7], m denotes meter and not the rack motion. A final constraint guarantees that the linkage is geometrically possible and will not be overextended. It is expressed as

[P4.8] $$\sqrt{s^2 - 0.0004} + \sqrt{t^2 - 0.0004} \ge \sqrt{h^2 + (c + m_{30})^2}$$

where m_{30} is the motion of the rack that is required for an outer steering angle θ_o of 30°.

A somewhat more difficult version of this problem is obtained by including the length c as one of the design parameters. An additional constraint must be imposed to ensure that the rack is long enough to produce the required steering angles. This constraint is expressed by

[P4.9] $$d - c \ge m + 0.02$$

10.4.5 Problem 5: Design of a Water Park Slide

Among the rides that we might see at a water amusement park is a slide. The rider sits on a slider to descend and is launched horizontally onto the surface of a pool from the end of the slide.

A Cartesian (x, y) system with x horizontal and y vertical is used to describe the equation of the slide. The top of the slide is at (x, y) equal to $(0, h)$, and the bottom is at (x, y) equal to $(L, 0)$. The rider is to leave the top with an initial horizontal velocity v_0 equal to 1.5 m/s, and is to exit horizontally at the bottom; therefore, the equation $y(x)$ representing the shape of the slide must satisfy the relations

[P5.1] $$y(0) = h; \quad y(L) = 0; \quad y'(0) = y'(L) = 0$$

in which y' denotes (dy/dx).

The equation that we shall use for the slide is given by

[P5.2] $$y/h = a\xi^4 + b\xi^3 + c\xi^2 + 1; \qquad \xi = x/L$$

The slide has a negative curvature on the left (that is, y″ equal to d^2y/dx^2 is negative), and it has a positive curvature on the right. At some neutral point, the curvature is zero. We shall choose the location of the neutral point through a parameter λ so that

[P5.3] $$y''(x) = 0 \text{ at } x = \lambda L \text{ (or at } \xi = \lambda)$$

and use λ as a parameter to vary the shape of the slide. The coefficients of Eq. [P5.2] that satisfy Eq. [P5.1] may be found in terms of λ.

The choice of λ is not arbitrary. To ensure that the ramp has a negative slope at all points except at the top and bottom, the second derivative must be negative when ξ is equal to 0, and it must be positive when ξ is equal to 1. The constraint that satisfies the requirement of negative slope is

[P5.4] $$(1/3) < \lambda < (2/3)$$

A complete description of the slide is obtained when h, λ, and L are specified. We shall set h equal to 5 m and treat λ and L as independent variable inputs.

Let us now consider the forces acting on a body of mass M as it descends the slide. The normal forces are perpendicular to the slide and consist of a weight component of the body and the reaction of the slide on the body. Tangential forces act along the tangent to the slide and consist of a weight component of the body and the retarding frictional force. A schematic of the problem showing the free-body diagram and the accelerations is provided in Fig. P5–1.

The weight components are the normal component ($Mg \cos \alpha$) and the tangential component ($Mg \sin \alpha$), in which α is the negative angle

[P5.5] $$\alpha = \arctan(y')$$

The normal force exerted on the body by the slide is denoted by N, and the tangential friction force is μN, with μ equal to 0.1 denoting the dynamic coefficient of friction.

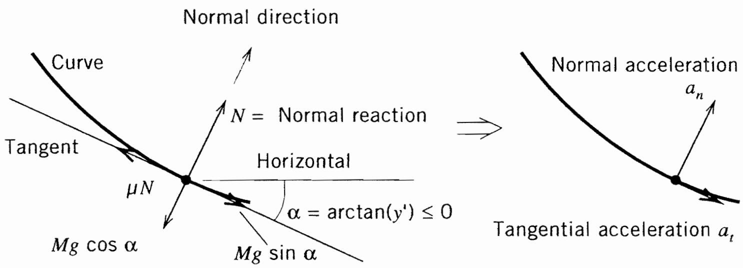

Figure P5–1 Schematic for the slide problem.

From Newton's Second Law, we obtain the normal equation of motion as

$$N - Mg\cos\alpha = Ma_n$$

Rearrangement and division by Mg yields

[P5.6] $$\Gamma = N/(Mg) = \cos\ \alpha + a_n/g$$

The quantity Γ is the number of "gees" experienced by the body. We shall restrict this value under ***frictionless*** conditions to be no more than 1.5.

The normal component of acceleration is

[P5.7] $$a_n = v^2/\rho$$

where v is the speed of the rider and ρ is the radius of curvature given by

[P5.8] $$\rho = [1 + (y')^2]^{1.5}/y''$$ put absolute value around this

The speed v with no friction is obtained from the energy conservation principle. It is found from

[P5.9] $$v^2 = (v_o)^2 + 2g(h - y); \qquad g = 9.81\ \text{m/s}^2; \qquad \mu = 0$$

Thus the constraint on Γ is

[P5.10] $$\Gamma = \cos\ \alpha + [(v_o)^2 + 2g(h - y)]/[\rho g] \leq 1.5$$

We now consider the tangential motion to determine the speed v when friction affects the motion. The tangential equation of motion is

$$a_t = d[v^2/2]/ds = g\ \sin\ \alpha - \mu N/M$$

replace sin(alpha) with sin(-alpha) in this equation and P5.11

where s is the arclength of the slide, and

$$(ds)^2 = (dx)^2 + (dy)^2$$

With N obtained from Eq. [P5.6], the ordinary differential equation and initial condition for the motion of the rider are

[P5.11] $$d(v^2)/dx = 2[g(\sin\alpha - \mu\cos\alpha) - \mu v^2/\rho]\sqrt{1+(y')^2};$$
$$v^2 = (v_0)^2 \text{ at } x = 0$$

The objective of the design is to define the slide that maximizes the exit speed v_e when the rider reaches the end of the slide. For the rider to reach the end of the slide, the solution for (v^2) cannot be negative for any x value from 0 to L.

An open-ended version of this problem is obtained by not restricting the shape of the slide to the function given in Eq. [P5.2]. References to the shape parameter λ are now meaningless, but negative slopes in the range $(0 < x < L)$ are still required. With the shape restriction removed, we must impose another constraint to ensure that the rider does not leave the slide. This constraint is

[P5.12] $$\Gamma = \cos\alpha + [(v_0)^2 + 2g(h-y)]/[\rho g] > 0$$

10.4.6 Problem 6: Design of a Ventilation System

This problem deals with a highly simplified ***ventilation system*** for a rectangular region with two openings. The geometry of the region is shown in Fig. P6–1.

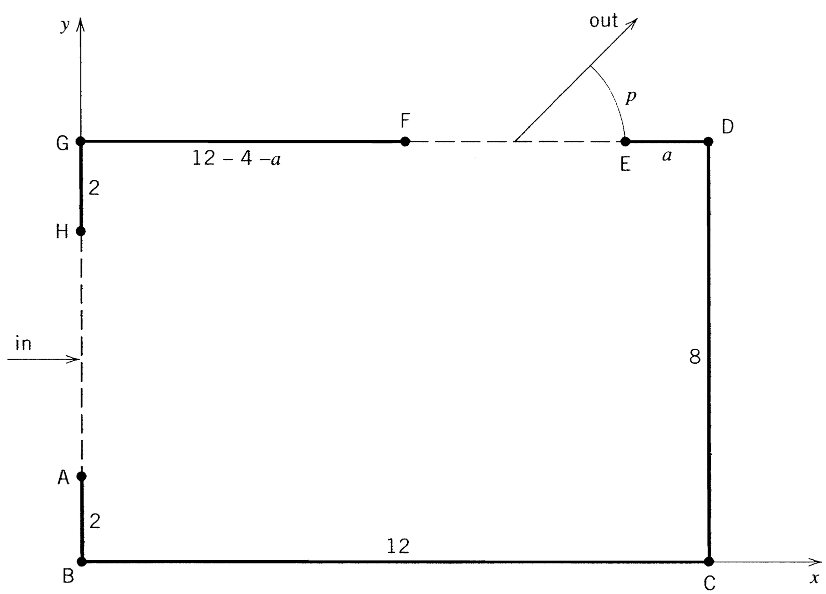

Figure P6–1 Schematic of the problem.

Dimensions in arbitrary length units are shown for each solid boundary. An exhaust fan pulls air out of the opening EF, and air enters through the other opening AH at an average nondimensional speed of 1. It is assumed that the inflow and outflow are in fixed directions and that the outflow is directed at an angle p to the line ED.

The model for the system is in terms of a stream function ψ, which is related to the x component u and the y component v of the velocity by

[P6.1] $$u = \psi_y; \qquad v = -\psi_x$$

The governing equation for the flow in the region is

[P6.2] $$\nabla^2\psi = 0$$

The boundary conditions for the problem are as follows.

[P6.3a] $$\psi = 0 \text{ on ABCDE}$$

[P6.3b] $$\psi_y = -\psi_x \cot p \text{ between E and F}$$

[P6.3c] $$\psi = \text{constant} = 4 \text{ on FGH}$$

[P6.3d] $$\psi_x = 0 \text{ between H and A}$$

The problem is to be solved on a grid with x and y spacings equal to 1. The speed U of the flow at all points not on the boundary is to be found from

[P6.4] $$U^2 = u^2 + v^2$$

The objective of the problem is to choose the angle p and the distance a so that the population standard deviation of U at the interior points of the field is minimized. If possible, provide plots of the streamlines (ψ = constant) to show the flow pattern.

The constraints for the problem are simple. The distance a must be at least 2 and no more than 6. The angle p must fall in the range ($45° \leq p \leq 135°$).

10.4.8 Problem 8: Design of a Rocket Launch Configuration

A small rocket has an initial total mass m_0 of 300 kg, which includes 180 kg of propellant. The rocket is to be launched from sea level at an initial angle γ_0 to the horizontal. After launch, the axis of the rocket is aligned with the flight path so that both the thrust T and the drag D are tangent to the flight path. The objective of the design is to configure the launch parameters so that the rocket has maximum (horizontal) range.

The equations of motion for the rocket are given in terms of a Cartesian (x, y) system, with x [m] denoting the horizontal (positive in the flight direction) and y [m] denoting the vertical (positive upward). A free-body diagram is given in Fig. P8–1.

Let u and v be the x and y components, respectively, of the rocket's velocity, and let t [s] denote time. Then we use the definition of velocity, Newton's Second Law, and the free-body diagram to write the governing system of ordinary differential equations as

[P8.1a] $$dx/dt = u; \qquad dy/dt = v$$

[P8.1b] $$du/dt = [(T - D)\cos\gamma]/m$$

[P8.1c] $$dv/dt = [(T - D)\sin\gamma]/m - g; \qquad g = 9.81 \text{ m/s}^2$$

The flight path angle γ in Eq. [P8.1b, c] is given by

[P8.2] $$\gamma = \tan^{-1}(v/u); \qquad -\pi/2 \text{ radian} < \gamma < \pi/2 \text{ radian}$$

To consider the other quantities (except the drag D) in those equations, we must first look at the consumption rate of the propellant.

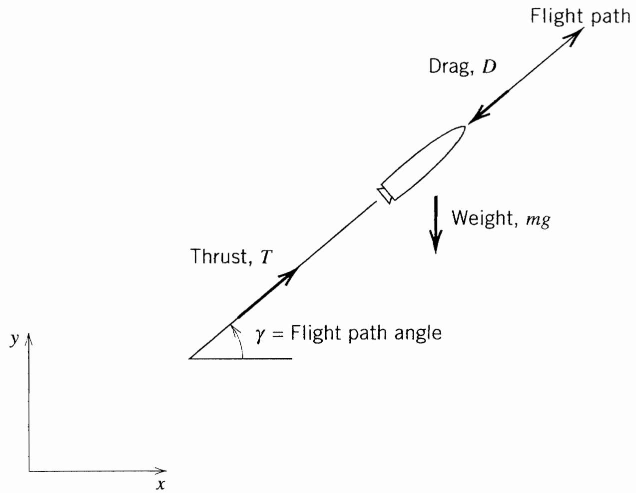

Figure P8–1 Free-body diagram for the rocket.

We shall assume that the propellant consumption rate q [kg/s] can be varied linearly with time from the instant of launch ($t = 0$) to the burnout time t_b (at which the propellant is completely consumed). The model for the consumption rate is

[P8.3] $$q = q_0 + (q_b - q_0)(t/t_b); \quad 0 \leq t \leq t_b$$

where q_0 [kg/s] is the initial rate at launch and q_b [kg/s] is the final rate at burnout. The consumption rates are constrained according to

[P8.4] $$7.5 \text{ kg/s} \leq q_0, q_b \leq 12.5 \text{ kg/s}$$

The mass of propellant that is consumed at time τ less than or equal to t_b is

[P8.5] $$m_p(\tau) = \int_0^\tau q \, dt; \qquad 0 \leq \tau \leq t_b$$

The burnout time t_b may therefore be found from

[P8.6] $$m_p(t_b) = 180 \text{ kg} = \text{total propellant mass}$$

and the instantaneous mass m [kg] of the rocket is given by

[P8.7] $$m = \begin{cases} 300 \text{ kg} - m_p(t); & 0 \leq t \leq t_b \\ 120 \text{ kg}; & t > t_b \end{cases}$$

The thrust T is equal to qv_e, where v_e is the exhaust speed of the propellant relative to the rocket and is equal to 500 m/s, as long as the propellant lasts. Thus

[P8.8] $$T = \begin{cases} q(500 \text{ m/s}); & 0 \leq t \leq t_b \\ 0; & t > t_b \end{cases}$$

The drag D [N] is given by

[P8.9] $$D = (0.1 \text{ N} \cdot \text{s}^2/\text{m}^2)(u^2 + v^2)(\rho/\rho_s)/\beta$$

The term (ρ/ρ_s) is a correction for the atmospheric density ρ at altitude y relative to the density ρ_s at sea level. A model for the density ratio is

[P8.10] $$\rho/\rho_s = (1 - [2.25 \times 10^{-5} \text{ m}^{-1}]y)^{4.25}$$

The quantity β is the Prandtl–Glauert compressibility correction as the Mach number M of the rocket becomes appreciable. The expression for β is

[P8.11] $$\beta = \sqrt{1 - M^2}; \quad M^2 = (u^2 + v^2)/c^2$$

in which c is the speed of sound in the atmosphere. For the altitudes that the rocket will attain, c may be taken as a constant equal to 340 m/s.

The range R [m] is the horizontal distance that the rocket flies before returning to sea level ($y = 0$). The objective of the design is to choose the initial flight path angle γ_0 and the propellant consumption rates q_0 and q_b so that the range is maximized. A table and a plot for the flight path of the rocket should also be provided; these should indicate when burnout occurs.

In solving the problem, care should be taken to obtain an accurate solution in the early phase of the flight, because the flight path angle may change rapidly during that period. The discontinuity in the thrust T and the discontinuous rate of change of the mass m at burnout must also be treated carefully.