# Solving boundary-value problems for systems of hyperbolic conservation laws with rapidly varying coefficients 

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Abstract<br>Solving boundary-value problems for systems of hyperbolic conservation laws with rapidly varying coefficients<br>by Darryl H. Yong<br>Chair of Supervisory Committee<br>Professor Jirair Kevorkian<br>Applied Mathematics

We study how boundary conditions affect the multiple-scale analysis of hyperbolic conservation laws with rapid spatial fluctuations. The most significant difficulty occurs when one does not have enough boundary conditions to solve consistency conditions. We show how to overcome this missing boundary condition difficulty for both linear and nonlinear problems through the recovery of boundary information. We introduce two methods for this recovery (multiple-scale analysis with a reduced set of scales, and a combination of Laplace transforms and multiple scales) and show that they are roughly equivalent. We also show that the recovered boundary information is likely to contain secular terms if the initial conditions are nonzero. However, for the linear problem we demonstrate how to avoid these secular terms to construct a solution that is valid for all time. For nonlinear problems, we argue that physically relevant problems do not exhibit the missing boundary condition difficulty.

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## NOTATION USED

$\epsilon$
General small parameter satisfying $0<\epsilon \ll 1$
$x^{*}=x / \epsilon \quad$ "Fast" or compressed spatial variable
$\tilde{x}=\epsilon X \quad$ Stretched spatial variable
$\hat{x}=\epsilon^{2} x \quad$ Very stretched spatial variable
$t^{*}=t / \epsilon \quad$ Fast temporal variable
$\tilde{t}=\epsilon t \quad$ Slow temporal variable
$\hat{t}=\epsilon^{2} t \quad$ Very slow temporal variable
$A, \mathcal{N} \quad$ Uppercase roman and script letters represent matrices
$\mathbf{u} \quad$ Bolded, roman letters represent vectors
$a_{12}, u_{1} \quad$ Numerical subscripts represent elements of vectors and matrices
$u_{t} \quad$ Non-numerical subscripts represent partial derivatives
$\mathbf{u}^{(2)} \quad$ Numerical superscripts in parenthesis represent terms in an asymptotic series Underlined variables are independent of $x^{*}$ Angled brackets represent averages with respect to $x^{*}$ (see Appendix A)
$\{a\} \quad$ Braces represent zero-average quantities with respect to $x^{*}$ (see Appendix A)

【a】 Double brackets represent zero-average integrals of $\{a\}$ (see Appendix A)
$\mathcal{O}\left(\epsilon^{n}\right) \quad$ "Big-O" Notation

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Dear Heavenly Father, the creator of all things and the keeper of all knowledge and wisdom, I pray that my work pleases and glorifies you.

## Chapter 1

## INTRODUCTION

The goal of our research is to apply the multiple-scale method to boundary-value problems for hyperbolic conservation laws with rapidly fluctuating quantities. We are interested in the effects that boundary conditions have on the multiple-scale analysis of these problems. In this chapter, we discuss hyperbolic conservation laws, why we use the method of multiple scales, and hint at the difficulties that boundary conditions pose.

### 1.1 Hyperbolic conservation laws and the theory of homogenization

Hyperbolic conservation laws, which typically arise from physical principles such as the conservation of energy or momentum, govern the propagation of information at finite speeds through some medium. The governing equations for shallow water waves and acoustics are canonical examples.

When properties of the medium being modeled have rapid spatial variations, the governing hyperbolic conservation laws typically exhibit these variations in their coefficients. For example, wave propagation in a bubbly liquid [7,25], and gravity waves in a channel with a rough bottom [28] are modeled using hyperbolic conservation laws with rapid spatial fluctuations. If these fluctuations are large (for example, the amplitude of fluctuations being comparable to its average value), regular perturbation techniques cannot be used. In such situations, the exact solution to the conservation laws can-
not be derived analytically. Instead, one seeks to determine the qualitative effects of these rapid spatial fluctuations (often dispersion or diffusion), or to find "effective" equations that describe the average behavior of the system.

What we have just described is one facet of the theory of homogenization. In short, the theory of homogenization attempts to describe the macroscopic properties of objects that have microscopic details. Often, one seeks to replace a very detailed mathematical description of a physical problem with a formulation that is mathematically simpler, yet still able to account for those details that have been "averaged" or "homogenized."

The literature in this area is vast and encompasses many disciplines of mathematics. Systematic treatment of the theory of homogenization began with the work of Bensoussan, Lions, and Papanicolaou in 1978, although the application of multiple scales to the theory of homogenization was anticipated by J. B. Keller [5]. Another standard reference in this field is Bakhvalov and Panasenko [3]. For a list of specific applications of homogenization techniques, see Kevorkian and Bosley [19].

### 1.2 About multiple-scale analysis

The basic assumption of the method of multiple scales is that the system being studied exhibits features at scales that are different enough that the features at each scale are somehow weakly correlated. For example, although the tiny bubbles in a liquid may cause minute variations in the propagation a wave through that liquid, our intuition tells us that the overall shape, speed and direction of the wave is not affected by the bubbles. Mathematically, this assumption means that the spatial scales $x^{*}=x / \epsilon, x$, $\tilde{x}=\epsilon x$, and $\hat{x}=\epsilon^{2} x$, with $0<\epsilon \ll 0$, are independent from one another even though they are linearly related to one another; we can therefore consider partial derivatives with respect to each of these spatial scales. Similarly, we argue that the temporal
scales $t, t^{*}=t / \epsilon$, and $\tilde{t}=\epsilon t$ are independent as well.
This notion of the separated scales is precisely why the method of multiple scales is naturally suited for problems whose media have properties with rapid spatial fluctuations. In fact, the method of multiple scales was the method of choice for Bensoussan, Lions, and Papanicolaou [5]. For a complete reference on the method of multiple scales, we refer the reader to the book by Kevorkian and Cole [20].

Another important assumption in our analysis is that the desired unknown quantities can be represented in an asymptotic expansion-in other words, that the unknown quantities can be approximated by a series of terms in some appropriate limit, usually $\epsilon \rightarrow 0$. The asymptotic expansion does not have to converge (and often is most useful when it does not), but each term in the asymptotic expansion must be smaller than the previous term in the region of validity of the expansion. For a thorough development of these ideas, we refer the reader to Chapter 1 of Murray's book [26].

### 1.3 Problem of the missing boundary conditions

Even though almost all physically relevant problems involve boundaries of some sort and therefore necessitate the enforcement of boundary conditions, there are many instances in which boundary conditions complicate matters considerably. For example, although it is more convenient to consider an infinite tank of shallow water, such a tank cannot exist. Kevorkian and Bosley have worked out the initial-value problem for a general system of hyperbolic conservation laws with rapid spatial fluctuations using multiple-scale analysis [19]. They do not discuss problems with both initial and boundary conditions because of a mathematical difficulty with "missing boundary conditions." In this section, we describe two ways this problem can occur.

In [30], Santosa and Symes use Bloch wave expansions to derive an effective equation for wave propagation in a periodic composite material. For the one-dimensional
linear wave equation

$$
\begin{equation*}
\rho\left(x^{*}\right) u_{t t}-\left(k\left(x^{*}\right) u_{x}\right)_{x}=0, \tag{1.1}
\end{equation*}
$$

with $\rho$ and $k$ as the density and bulk modulus and $x^{*}=x / \epsilon$ as the "fast" spatial variable $(\epsilon \ll 1)$, they find that the homogenized behavior has the governing equation

$$
\frac{\partial^{2} w_{1}^{(h)}}{\partial t^{2}}-\alpha \frac{\partial^{2} w_{1}^{(h)}}{\partial x^{2}}+\epsilon^{2} \beta \frac{\partial^{4} w_{1}^{(h)}}{\partial x^{4}}+\mathcal{O}\left(\epsilon^{3}\right)=0
$$

as $\epsilon \rightarrow 0$. (This equation is also derived in [19].) Here, $w_{1}^{(h)}$ is one of the homogenized characteristic dependent variables (the other dependent variable $w_{2}^{(h)}$ has a similar governing equation), and the constants $\alpha$ and $\beta$ are complicated expressions related to $\rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$. The most significant part of this homogenized equation is the fourth partial derivative with respect to the spatial dimension, indicating dispersion (waves speed varies with wavenumber). For an initial-value problem the fourth partial derivative poses no complications, but for a problem with boundaries one needs more boundary information than is known from the outset of solving (1.1). Santosa and Symes note that, "in the presence of boundaries, none of what is discussed up to now is valid, some other approach will be necessary." [30]

The difficulty of missing boundary conditions manifests itself in a different manner through the method of multiple scales. When using the method of multiple scales, one typically encounters consistency conditions (sometimes also known as solvability conditions) that must be satisfied to construct an asymptotically valid answer (to avoid secular terms). These consistency conditions can be of a different class of partial differential equations (PDEs) as the original problem and therefore require more boundary and initial conditions than can be specified for the original problem without creating an ill-posed problem.

### 1.4 Proposed work

In this thesis, we present two methods for analyzing systems of hyperbolic conservation laws with rapid spatial fluctuations. The first is the usual multiple-scale method (Chapter 4), and the second is a combination of multiple-scale analysis with Laplace transforms (Chapter 5). Using both methods, we highlight the missing boundary condition difficulty and show how to overcome it through the recovery of boundary information.

Before these main chapters, we first describe how to solve systems of hyperbolic conservation laws without rapid spatial fluctuations when boundary conditions are specified. The results obtained in Chapter 2 will give us some insights to the problems with rapid spatial fluctuations.

In Chapter 3, we recount how to write a general system of hyperbolic conservation laws in a standard form. We also discuss the boundedness of solutions to a linear problem and how initial conditions must be carefully chosen to avoid solutions that depend on the fast temporal scale, $t^{*}=t / \epsilon$.

## Chapter 2

## INITIAL-BOUNDARY VALUE PROBLEMS FOR A PAIR OF CONSERVATION LAWS

Before we study hyperbolic conservation laws with rapidly fluctuating coefficients, we first analyze initial-boundary value problems for hyperbolic conservation laws without such fluctuations. The goal of this section is to identify how boundary conditions affect the multiple-scale solution procedure.

We begin this chapter by studying the one-dimensional wavemaker problem for shallow water flow. Next, we generalize our results to a general nonlinear pair of hyperbolic conservation laws. We do not consider systems of three or more conservation laws to avoid the possibility of resonant interactions between the dependent variables for certain periodic initial conditions [17].

One way to obtain closed form solutions to a general system of nonlinear conservation laws is to perturb about a constant steady state, reducing a potentially nonlinear problem to a linear one. In these situations, the appropriate strategy for an initialvalue problem is to use slow temporal scales (for example, $\tilde{t}=\epsilon t$ where $0<\epsilon \ll 1$ ) to capture any nonlinear effects (Section 8.3.2 of [18]). For a signaling problem, in which boundary conditions represent signals that propagate into an initially quiescent medium, the correct strategy is to use stretched spatial scales, such as $\tilde{x}=\epsilon X$ (Section 6.2 .4 of [20]). For a problem with both initial and boundary conditions, we anticipate that the correct solution procedure involves both slow temporal scales and stretched spatial scales.

### 2.1 Wavemaker problem for shallow water flow

To illustrate the main ideas of this chapter, let us consider a wavemaker problem for one-dimensional shallow water waves. Let $h(x, t ; \epsilon)$ and $u(x, t ; \epsilon)$ be the height and velocity of water in a one-dimensional tank. The governing equations,

$$
\begin{align*}
h_{t}+(u h)_{x} & =0  \tag{2.1a}\\
u_{t}+h_{x}+u u_{x} & =0 \tag{2.1b}
\end{align*}
$$

are derived from physical principles in Section 3.2 of [18]. Because we are more interested in how the solution procedure is affected by boundaries rather than obtaining highly accurate solutions, we have ignored higher-order correction terms in (2.1b) that account for the motion in the vertical direction [20].

Equations (2.1) have been normalized so that the resting state of the water corresponds to $u=0$ and $h=1$. Suppose that our tank is semi-infinitely long with a wavemaker situated near the origin at $x_{p}=\epsilon p(t)$, where $0<\epsilon \ll 1$ is our usual small parameter. The wavemaker introduces the boundary condition

$$
\begin{equation*}
u(\epsilon p(t), t ; \epsilon)=\epsilon p^{\prime}(t) \tag{2.2}
\end{equation*}
$$

In addition, let us prescribe some initial height and velocity perturbations:

$$
\begin{align*}
& h(x, 0 ; \epsilon)=1+\epsilon g(x)  \tag{2.3a}\\
& u(x, 0 ; \epsilon)=\epsilon v(x) \tag{2.3b}
\end{align*}
$$

Figure 2.1 shows a picture of our one-dimensional tank.
Notice that in equation (2.2) we implicitly assume that the wavemaker does not move very much. This assumption allows us to use a Taylor series expansion to replace a moving boundary problem with a fixed boundary problem:

$$
\begin{equation*}
u(\epsilon p(t), t ; \epsilon)=u(0, t ; \epsilon)+\epsilon u_{x}(0, t ; \epsilon) p(t)+\frac{1}{2} \epsilon^{2} u_{x x}(0, t ; \epsilon) p^{2}(t)+\mathcal{O}\left(\epsilon^{3}\right)=\epsilon p^{\prime}(t) \tag{2.4}
\end{equation*}
$$



Figure 2.1: Wavemaker problem for one-dimensional shallow water waves.
as $\epsilon \rightarrow 0$. The solution domain for our problem is now the quarter space $x>0$ and $t>0$.

We assume that the unknown functions have the asymptotic expansions ${ }^{1}$

$$
\begin{align*}
& u(x, t ; \epsilon)=\epsilon u^{(1)}(x, \tilde{x}, t, \tilde{t})+\epsilon^{2} u^{(2)}(x, \tilde{x}, t, \tilde{t})+\mathcal{O}\left(\epsilon^{3}\right)  \tag{2.5a}\\
& h(x, t ; \epsilon)=1+\epsilon h^{(1)}(x, \tilde{x}, t, \tilde{t})+\epsilon^{2} h^{(2)}(x, \tilde{x}, t, \tilde{t})+\mathcal{O}\left(\epsilon^{3}\right) \tag{2.5b}
\end{align*}
$$

as $\epsilon \rightarrow 0$. To capture the nonlinear behavior of (2.1) we include the slow scales $\tilde{x}=\epsilon X$ and $\tilde{t}=\epsilon t$. Another reason to include these additional scales is that without them, we would not be able to avoid secular terms that cause our solution to grow linearly in time and space. (We discuss secular terms more completely on the next page.)

[^0]
### 2.1.1 $\mathcal{O}(\epsilon)$ system

Now we plug (2.5) into (2.1) and collect all terms proportional to $\epsilon$, obtaining the $\mathcal{O}(\epsilon)$ system of equations,

$$
\begin{align*}
& h_{t}^{(1)}+u_{x}^{(1)}=0  \tag{2.6a}\\
& u_{t}^{(1)}+h_{x}^{(1)}=0, \tag{2.6b}
\end{align*}
$$

which are subject to the initial and boundary conditions

$$
\begin{align*}
h^{(1)}(x, \tilde{x}, 0,0) & =g(x),  \tag{2.7a}\\
u^{(1)}(x, \tilde{x}, 0,0) & =v(x), \quad \text { and }  \tag{2.7b}\\
u^{(1)}(0,0, t, \tilde{t}) & =p^{\prime}(t) . \tag{2.7c}
\end{align*}
$$

We solve (2.6) by introducing the characteristic independent variables, $\xi=x-t$ and $\eta=x+t$, along with the characteristic dependent variables, $R^{(i)}=h^{(i)}+u^{(i)}$ and $L^{(i)}=h^{(i)}-u^{(i)}$. We find that

$$
R_{\eta}^{(1)}=L_{\xi}^{(1)}=0,
$$

which implies that $R^{(1)}$ represents waves traveling to the "right" in the $x-t$ plane, and $L^{(1)}$ represents waves traveling to the "left." We cannot determine anything about the $\tilde{x}$ - and $\tilde{t}$-dependencies of $R^{(1)}$ and $L^{(1)}$ until we consider the equations arising at the next order of $\epsilon$. The initial and boundary conditions (2.7) will be used later.

### 2.1.2 $\mathcal{O}\left(\epsilon^{2}\right)$ system

Collecting terms proportional to $\epsilon^{2}$ yields

$$
\begin{array}{r}
h_{\tilde{t}}^{(1)}+h_{t}^{(2)}+u_{\tilde{x}}^{(1)}+\left[u^{(1)} h^{(1)}+u^{(2)}\right]_{x}=0 \\
u_{\tilde{t}}^{(1)}+u_{t}^{(2)}+h_{\tilde{x}}^{(1)}+h_{x}^{(2)}+u^{(1)} u_{x}^{(1)}=0,
\end{array}
$$

which can be written in terms of characteristic independent and dependent variables as

$$
\begin{align*}
& R_{\eta}^{(2)}=-\frac{1}{2}\left[R_{\tilde{t}}^{(1)}+R_{\tilde{x}}^{(1)}+\frac{3}{4} R^{(1)} R_{\xi}^{(1)}\right]+\frac{1}{8}\left(L^{(1)} L_{\eta}^{(1)}+R^{(1)} L_{\eta}^{(1)}+L^{(1)} R_{\xi}^{(1)}\right)  \tag{2.8a}\\
& L_{\xi}^{(2)}=\frac{1}{2}\left[L_{\tilde{t}}^{(1)}-L_{\tilde{x}}^{(1)}-\frac{3}{4} L^{(1)} L_{\eta}^{(1)}\right]+\frac{1}{8}\left(R^{(1)} R_{\xi}^{(1)}+R^{(1)} L_{\eta}^{(1)}+L^{(1)} R_{\xi}^{(1)}\right) \tag{2.8b}
\end{align*}
$$

The quantity in the square brackets in (2.8a) is independent of $\eta$, so simply integrating (2.8a) with respect to $\eta$ will result in terms proportional to $\eta$, examples of so-called secular terms. ${ }^{2}$ Secular terms should be avoided because they limit the region of validity of our asymptotic expansion; for large $\eta$, the terms in the asymptotic expansion outgrow their assigned orders of magnitude. Likewise, the quantity inside the square brackets in (2.8b) must be set to zero. Therefore, to avoid all secular terms, the following pair of consistency conditions (sometimes also known as solvability conditions),

$$
\begin{align*}
R_{\tilde{t}}^{(1)}+R_{\tilde{x}}^{(1)}+\frac{3}{4} R^{(1)} R_{\xi}^{(1)} & =0  \tag{2.9a}\\
L_{\tilde{t}}^{(1)}-L_{\tilde{x}}^{(1)}-\frac{3}{4} L^{(1)} L_{\eta}^{(1)} & =0 \tag{2.9b}
\end{align*}
$$

must be satisfied, subject to the conditions

$$
\begin{align*}
& \left.L^{(1)}\right|_{t=0}=g(x)-v(x)  \tag{2.10a}\\
& \left.R^{(1)}\right|_{t=0}=g(x)+v(x), \quad \text { and }  \tag{2.10b}\\
& \left.R^{(1)}\right|_{x=0}-\left.L^{(1)}\right|_{x=0}=2 p^{\prime}(t) \tag{2.10c}
\end{align*}
$$

Notice that the boundary condition (2.10c) now involves a linear combination of $L^{(1)}$ and $R^{(1)}$. (To obtain more accurate solutions, a correction term should be added to (2.1b) which changes (2.9) from quasi-linear first-order partial differential equations to a pair of Korteweg-de Vries equations [18]).

[^1]The key to solving (2.9) is choosing the correct scales for $R^{(1)}$ and $L^{(1)}$. Because $L^{(1)}$ represents waves that are traveling to the left in the $x-t$ plane (towards the wavemaker), these left-going waves are primarily defined by the initial condition (2.10a) and don't interact with the boundary condition until they meet the wavemaker. Therefore, we should choose scales that are appropriate for an initial-value problem; in other words, we let $L^{(1)}=L^{(1)}(\eta, \tilde{t})$.

The situation for $R^{(1)}$ is a little more complicated as there are some outgoing waves that are influenced solely by the initial height and velocity perturbation, and there are some that are caused by the wavemaker. To make this distinction clear, we separate the solution domain, $x>0$ and $t>0$, into two regions by introducing a positive, monotone increasing function $J(t)$ with $J(0)=0$ so that $x=J(t)$ is the interface between the two regions. Let's choose Region A to have $x=0$ and $x=J(t)$ as its boundaries, Region B to have $t=0$ as one of its boundaries. See Figure 2.2.


Figure 2.2: Solution domain divided into two regions.

We denote $R^{(A)}$ and $R^{(B)}$ for $R^{(1)}$ in regions A and B , respectively. As Region B is ahead of the interface, the water there does not yet feel the influence of the boundary. Therefore, we should choose the scales that are appropriate for an initial-value problem
in Region B ; in other words, we use let $R^{(B)}=R^{(B)}(\xi, \tilde{t})$. In Region A , we will allow $R^{(A)}$ to depend on both $\tilde{x}=\epsilon X$ and $\tilde{t}$ by defining $R^{(A)}=R^{(A)}(\xi, \tilde{x}, \tilde{t})$. To summarize,

$$
\begin{aligned}
& L^{(1)}=L^{(1)}(\eta, \tilde{t}), \\
& R^{(1)}= \begin{cases}R^{(A)}(\xi, \tilde{x}, \tilde{t}) & \text { if } x<J(t) \\
R^{(B)}(\xi, \tilde{t}) & \text { if } x>J(t)\end{cases}
\end{aligned}
$$

It can be shown that if $R^{(A)}$ is not allowed to depend on $\tilde{x}$, there will not be enough degrees of freedom to satisfy the initial and boundary conditions in (2.10).

The governing equations for $L^{(1)}$ and $R^{(B)}$ are

$$
\begin{aligned}
R_{\tilde{t}}^{(B)}+\frac{3}{4} R^{(B)} R_{\xi}^{(B)} & =0 \\
L_{\tilde{t}}^{(1)}-\frac{3}{4} L^{(1)} L_{\eta}^{(1)} & =0,
\end{aligned}
$$

since they do not depend on $\tilde{x}$. These first-order quasilinear partial differential equations are easily solved using the method of characteristics. Keeping in mind that the initial conditions are given in (2.10a) and (2.10b), their solutions are

$$
\begin{align*}
& R^{(B)}(\xi, \tilde{t})=g(\bar{x})+v(\bar{x}), \text { where } \bar{x} \text { solves } \xi=\frac{3}{4} \tilde{t}(g(\bar{x})+v(\bar{x}))+\bar{x}  \tag{2.11a}\\
& L^{(1)}(\eta, \tilde{t})=g(\bar{x})-v(\bar{x}), \text { where } \bar{x} \text { solves } \eta=-\frac{3}{4} \tilde{t}(g(\bar{x})-v(\bar{x}))+\bar{x} . \tag{2.11b}
\end{align*}
$$

Without knowing more about the specific functions $g(x)$ and $v(x)$, these solutions can only be expressed as implicitly defined functions.

Once $L^{(1)}$ is known, the boundary condition (2.10c) becomes

$$
R^{(A)}(-t,-\tilde{t}, \tilde{t})=L^{(1)}(t, \tilde{t})+2 p^{\prime}(t)
$$

Using the method of characteristics again, we can now obtain an implicitly defined solution for $R^{(A)}$ :

$$
\begin{equation*}
R^{(A)}(\xi, \tilde{x}, \tilde{t})=f(-\bar{\xi}, \tilde{t}-\tilde{x}), \text { where } \bar{\xi} \text { solves } \xi=\frac{3}{4} \tilde{x} f(-\bar{\xi}, \tilde{t}-\tilde{x})+\bar{\xi} \tag{2.12}
\end{equation*}
$$

and $f(t, \tilde{t})=L^{(1)}(t, \tilde{t})+2 p^{\prime}(t)$.
Because (2.9a) and (2.9b) are first-order quasilinear partial differential equations (sometimes referred to as inviscid Burgers' equations), they admit solutions with shocks. Even when the initial conditions are continuous, wave steepening can lead to shocks forming at later times. When a shock forms, we must turn to the integral formulation of the original conservation laws instead of using (2.9) to determine the shock trajectory. This is because a quasilinear-first order equation like (2.9) can be manipulated to predict a variety of different shock speeds. Since (2.9) is an artificial equation that is not based on physical principles, we cannot expect it to predict the correct shock trajectory in its current form.

The true governing equations for shallow water flow exhibiting the proper flux and conserved quantities, are

$$
\begin{align*}
h_{t}+(u h)_{x} & =0  \tag{2.13a}\\
(u h)_{t}+\left(u^{2} h+\frac{h^{2}}{2}\right)_{x} & =0 . \tag{2.13b}
\end{align*}
$$

Therefore, the correct shock speed is governed by the pair of equations

$$
\begin{align*}
\frac{d s}{d t}[h]_{-}^{+} & =[u h]_{-}^{+}  \tag{2.14a}\\
\frac{d s}{d t}[u h]_{-}^{+} & =\left[u^{2} h+\frac{h^{2}}{2}\right]_{-}^{+} \tag{2.14b}
\end{align*}
$$

where $s(t)$ is the shock trajectory and the notation $[\cdot]_{-}^{+}$denotes the value of the jump of a quantity across its discontinuity. For example, if the shock occurs for $R^{(1)}$, plugging the expansion (2.5) into (2.14) gives the ordinary differential equation,

$$
\begin{equation*}
\frac{d K}{d \tilde{\eta}}=\frac{\left[\frac{3}{8}\left(R^{(1)}\right)^{2}\right]_{-}^{+}}{\left[R^{(1)}\right]_{-}^{+}}=\frac{3}{8}\left[R^{(A)}+R^{(B)}\right]_{\xi=K(\tilde{\eta})} \tag{2.15}
\end{equation*}
$$

which governs the shock trajectory, here written as $\xi=x-t=K(\tilde{\eta})$, where $\tilde{\eta}=\epsilon \eta$. The details of this derivation are given in Section 6.2.4 of [20]. (In this situation, it
just happens that the consistency condition (2.9a) produces the correct shock speed, but this may not be true in general.)

Now the only remaining task is to find $J(t)$, the trajectory of the interface between Regions $A$ and $B$. When the initial velocity perturbation (2.3b) exactly matches the water velocity imposed by the wavemaker in (2.2), the interface $J(t)$ is simply the characteristic emanating from the origin, which we know from the form of the characteristic independent variables is $x=t$. In more realistic situations, the initial condition and boundary condition will not match exactly and a shock or a fan will result.

The case of a fan is exemplified in a dam-breaking problem in Section 4.3.4 of [18]. In this situation, the solution domain should be divided into three regions: Region A (water under the influence of the wavemaker), Region B (water under the influence of the initial height and velocity perturbations), and a fan region between Regions $A$ and $B$.

In the case when a shock forms, the trajectory of the interface between Regions A and $B$ is the shock itself. Rewriting (2.15) using physical independent variables, we obtain

$$
\begin{equation*}
\frac{d J}{d t}=\left.\frac{1+\epsilon \frac{3}{8}\left(R^{(A)}+R^{(B)}\right)}{1-\epsilon \frac{3}{8}\left(R^{(A)}+R^{(B)}\right)}\right|_{x=J(t)} . \tag{2.16}
\end{equation*}
$$

(All occurrences of $\tilde{t}$ and $\tilde{x}$ in the right hand side of $(2.16)$ must now be written in terms of $t$ and $x$.) In most situations, (2.16) will be a nonlinear ordinary differential equation and must be solved numerically subject to the initial condition $J(0)=0$. If $\epsilon$ is small, we can approximate the shock trajectory with a straight line by using the initial slope as the slope for the entire line:

$$
J(t) \approx\left(\frac{1+\epsilon \frac{3}{4}\left(g(0)+p^{\prime}(0)\right)}{1-\epsilon \frac{3}{4}\left(g(0)+p^{\prime}(0)\right)}\right) t
$$

### 2.1.3 Numerical verification

We performed numerical calculations using the CLAWPACK software package [24] written by Randall J. LeVeque. We chose the initial and boundary conditions,

$$
\begin{aligned}
u(x, 0 ; \epsilon) & =0 \\
h(x, 0 ; \epsilon) & =1+\epsilon \frac{x}{x+1} \\
u(\epsilon p(t), t ; \epsilon) & =\epsilon p^{\prime}(t) \quad \text { with } \quad p(t)=\frac{t}{t+1},
\end{aligned}
$$

so that a shock forms because of a mismatch of the velocity at $x=t=0$ and no additional shocks will form at later times. We took $\epsilon=0.1$ and approximated the shock trajectory with a line as discussed above. We chose a spatial step size of 0.05 and used Godunov's method (no second-order corrections).

Figure 2.3 shows the numeric and analytic solutions five units of time after the initial height and velocity perturbations. The agreement between the two solutions is fairly good; the slight discrepancy in the tail of $u$ beyond the shock could be improved by a more accurate shock location. Notice that the analytic solution exhibits a discontinuity in the height and velocity at the shock, while the numeric solution presents a smooth solution because of numerical dissipation.

### 2.2 General pair of conservation laws

Consider now a pair of conservation laws written in differential form,

$$
\begin{equation*}
\mathbf{p}_{t}+\mathbf{q}_{x}=0 \tag{2.17}
\end{equation*}
$$

where the conserved quantity $\mathbf{p}$ and the flux $\mathbf{q}$ are two-component vectors. Let $\epsilon$ be a small, positive parameter: $0<\epsilon \ll 1$. We assume that each $p_{i}, q_{i}$ and $s_{i}$ is a function of the two dependent variables, $u_{1}$, and $u_{2}$ so that the conservation laws are spatially homogeneous.


Figure 2.3: Comparison of a numeric versus analytic solution of a wavemaker problem for the shallow water wave equations at $t=5$.

Besides shallow water flow, equation (2.17) can represent nearly isentropic gas dynamics (see Section 3.3.4 of [18]). With the addition of a source term on the right-hand side of (2.17), one could model glacier flow, chemical exchange processes, chromatography, sedimentation in glaciers and flow in a channel (see Chapter 3 of [32]). In this thesis, we restrict our attention to the cases without sources because they would obscure our goal of understanding problems with boundaries. A discussion of the initial-value problem for a class of conservation laws with source terms can be found in [21].

Evaluating derivatives with respect to $t$ and $x,(2.17)$ can be reduced to

$$
\begin{equation*}
\mathbf{u}_{t}+A(\mathbf{u}) \mathbf{u}_{x}=0 \tag{2.18}
\end{equation*}
$$

where

$$
A(\mathbf{u})=\left[\frac{\partial\left(p_{1}, p_{2}\right)}{\partial\left(u_{1}, u_{2}\right)}\right]^{-1}\left[\frac{\partial\left(q_{1}, q_{2}\right)}{\partial\left(u_{1}, u_{2}\right)}\right]
$$

is a product of two Jacobian matrices.
To make progress on an analytic solution to (2.18), we assume that there exists a constant steady state, $\mathbf{u}^{(0)}$, about which we construct a perturbation expansion:

$$
\begin{equation*}
\mathbf{u}(x, t ; \epsilon)=\mathbf{u}^{(0)}+\epsilon u_{O}^{()}(x, \tilde{x}, t, \tilde{t})+\mathbf{u}^{(2)}(x, \tilde{x}, t, \tilde{t})+\mathcal{O}\left(\epsilon^{3}\right) \tag{2.19}
\end{equation*}
$$

With this choice of scales, derivatives with respect to $x$ and $t$ become

$$
\begin{align*}
& \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial x}+\epsilon \frac{\partial}{\partial \tilde{x}}  \tag{2.20a}\\
& \frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t}+\epsilon \frac{\partial}{\partial \tilde{t}} \tag{2.20b}
\end{align*}
$$

We also expand the matrix $A(\mathbf{u})$ as

$$
\begin{equation*}
A(\mathbf{u})=A\left(\mathbf{u}^{(0)}+\epsilon \mathbf{u}^{(1)}+\cdots\right)=A^{(0)}+\epsilon A^{(1)}+\epsilon^{2} A^{(2)}+\mathcal{O}\left(\epsilon^{3}\right) \tag{2.21}
\end{equation*}
$$

where

$$
\begin{aligned}
A^{(0)} & =A\left(\mathbf{u}^{(0)}\right) \\
a_{i j}^{(1)} & =\frac{\partial a_{i j}}{\partial u_{m}}\left(\mathbf{u}^{(0)}\right) u_{m}^{(1)} \\
a_{i j}^{(2)} & =\frac{\partial a_{i j}}{\partial u_{m}}\left(\mathbf{u}^{(0)}\right) u_{m}^{(2)}+\frac{\partial^{2} a_{i j}}{\partial u_{m} \partial u_{n}}\left(\mathbf{u}^{(0)}\right) u_{m}^{(1)} u_{n}^{(1)} .
\end{aligned}
$$

(We adopt the summation convention for repeated indices on $m$ and $n$.)
We plug in the expansions (2.19) and (2.21) into (2.18), use the change of derivatives (2.20), and collect like powers of $\epsilon$ to obtain the governing equations for each order of $\mathbf{u}^{(i)}$.

### 2.2.1 $\mathcal{O}(\epsilon)$ system

The governing equations for $\mathbf{u}^{(1)}$ are

$$
\begin{equation*}
\mathbf{u}_{t}^{(1)}+A^{(0)} \mathbf{u}_{x}^{(1)}=0 \tag{2.22}
\end{equation*}
$$

The differential operator ${ }^{3}$ represented by the left hand side of (2.22) will govern the solution to every order of $\epsilon$. Therefore, for (2.18) to describe a hyperbolic system of conservation laws, the eigenvalues of $A^{(0)}$ must be real and distinct. We will also assume that one eigenvalue is positive and the other is negative-something that can be accomplished through an appropriate change of independent variables. Furthermore, the reason that we assumed the steady state $\mathbf{u}^{(0)}$ to be a constant is that we can only find a general solution to $(2.22)$ if $A^{(0)}$ is a constant matrix. Under these assumptions, we are able to diagonalize $A^{(0)}$ by defining

$$
\Lambda=P^{-1} A^{(0)} P=\left[\begin{array}{cc}
\lambda_{1} & 0 \\
0 & \lambda_{2}
\end{array}\right]
$$

Without loss of generality, let us choose $\lambda_{1}$ to be the positive eigenvalue and $\lambda_{2}$ to be the negative one. We define $w^{(i)}=P^{-1} u^{(i)}$ to be the characteristic dependent variables, and express (2.22) in these new variables, written out in component form:

$$
\begin{align*}
& w_{1 t}^{(1)}+\lambda_{1} w_{1 x}^{(1)}=0  \tag{2.23a}\\
& w_{2 t}^{(1)}+\lambda_{2} w_{2 x}^{(1)}=0 . \tag{2.23b}
\end{align*}
$$

These equations imply that $w_{1}^{(1)}$ is a function of $\xi=x-\lambda_{1} t$, and $w_{2}^{(1)}$ is a function of $\eta=x-\lambda_{2} t$. Since $\lambda_{1}$ is positive, that means that $w_{1}^{(1)}$ represents a wave that is traveling to the "right" in the $x$ - $t$ plane, and $w_{2}^{(1)}$ a wave to the "left."

As we saw in the shallow water flow example, the choice of stretched spatial scales and slow temporal scales for $w_{i}^{(1)}$ depends on the arrangement of the solution domain. If our solution domain is the quarter space, $x>0$ and $t>0$, then the appropriate choice of scales is $w_{1}^{(1)}=w_{1}^{(1)}(\xi, \tilde{x}, \tilde{t})$ and $w_{2}^{(1)}=w_{2}^{(1)}(\xi, \tilde{t})$. We anticipate defining $w_{1}^{(1)}$ as a multipart function representing waves originating from the initial condition

[^2]or waves originating from the boundary condition. The dependence of $\mathbf{w}^{(1)}$ on these slow scales cannot be determined until we consider the equations arising at the next order of $\epsilon$.

### 2.2.2 $\mathcal{O}\left(\epsilon^{2}\right)$ system

The equations governing $\mathbf{u}^{(2)}$ are

$$
\begin{equation*}
\mathbf{u}_{t}^{(2)}+\mathbf{u}_{\tilde{t}}^{(1)}+A^{(1)} \mathbf{u}_{x}^{(1)}+A^{(0)}\left(\mathbf{u}_{x}^{(2)}+\mathbf{u}_{\tilde{x}}^{(1)}\right)=0 \tag{2.24}
\end{equation*}
$$

which may be written using characteristic independent and dependent variables as

$$
\left[\begin{array}{l}
\left(\lambda_{1}-\lambda_{2}\right) w_{1 \eta}^{(2)}  \tag{2.25}\\
\left(\lambda_{2}-\lambda_{1}\right) w_{2 \xi}^{(2)}
\end{array}\right]+\mathbf{w}_{\tilde{t}}^{(1)}+\Lambda \mathbf{w}_{\tilde{\chi}}^{(1)}+P^{-1} A^{(1)} P\left[\begin{array}{l}
w_{1 \xi}^{(1)} \\
w_{2 \eta}^{(1)}
\end{array}\right]=0 .
$$

(We have used $\partial_{x} \rightarrow \partial_{\xi}+\partial_{\eta}$ and $\partial_{t} \rightarrow-\lambda_{1} \partial_{\xi}-\lambda_{2} \partial_{\eta}$ to get the equation above.) If the first component of $(2.25)$ is integrated with respect to $\eta$, we will obtain secular terms taking the form of $\eta$ premultiplying all terms independent of $\eta$. Therefore, all terms independent of $\eta$, like $w_{1}^{(1)}(\xi, \tilde{x}, \tilde{t})$, must be removed from the first component of (2.25). Likewise, all terms independent of $\xi$ must be removed from the second component. This leads to the consistency conditions

$$
\begin{align*}
w_{1 \tilde{t}}^{(1)}+\lambda_{1} w_{1 \tilde{x}}^{(1)}+\left(b_{11} p_{11}+c_{11} p_{21}\right) w_{1}^{(1)} w_{1 \xi}^{(1)} & =0  \tag{2.26a}\\
w_{2 \tilde{t}}^{(1)}+\left(b_{22} p_{12}+c_{22} p_{22}\right) w_{2}^{(1)} w_{2 \eta}^{(1)} & =0 \tag{2.26b}
\end{align*}
$$

where

$$
B=P^{-1} \frac{\partial A}{\partial u_{1}}\left(\mathbf{u}^{(0)}\right) P \quad \text { and } \quad C=P^{-1} \frac{\partial A}{\partial u_{2}}\left(\mathbf{u}^{(0)}\right) P
$$

As mentioned before, these two quasilinear equations admit solutions with shocks. When shocks form, one must return to the integral formulation of the original conservation laws to determine the correct shock trajectory.

### 2.3 Discussion

We now return to the original question posed at the beginning of this chapter: "What is the effect of boundaries on the multiple-scale analysis?" All of the problems solved in this chapter have the common feature that the $\mathcal{O}(\epsilon)$ solution can be decomposed into two waves, each traveling in a different direction (for example $L^{(1)}$ and $R^{(1)}$ for the wavemaker problem). The wave that is traveling towards the boundary condition is primarily determined by its initial condition, so it is the wave that travels away from the boundary that has the responsibility of satisfying the boundary condition. In these problems, we have used the stretched spatial scale $\tilde{x}$ to give this "outbound" wave the extra freedom to satisfy the boundary condition.

Let's take a closer look at exactly how the extra freedom is achieved. The effect of including $\tilde{x}$ first presents itself in the consistency conditions. For example, in the wavemaker problem, the consistency conditions for the wave traveling to the right are

$$
\begin{align*}
& R_{\tilde{t}}^{(A)}+R_{\tilde{x}}^{(A)}+\frac{3}{4} R^{(A)} R_{\xi}^{(A)}=0  \tag{2.27a}\\
& \text { and } \quad R_{\tilde{t}}^{(B)}+\frac{3}{4} R^{(B)} R_{\xi}^{(B)}=0 \tag{2.27b}
\end{align*}
$$

We don't include $\tilde{x}$ in (2.27b) because the water in Region B is primarily influenced by the initial conditions and the appropriate solution procedure for an initial-value problem is to add the slow temporal scale $\tilde{t}$ only. What is the difference between (2.27a) and (2.27b)?

First, we point out that once we convert (2.27a) and (2.27b) back to physical coordinates, they actually represent very similar equations. Let $f(\xi, \tilde{x}, \tilde{t})=R^{(1)}+$ $\epsilon R^{(2)}+\cdots$ in Region A and $g(\xi, \tilde{t})=R^{(1)}+\epsilon R^{(2)}+\cdots$ in Region B represent the wave traveling to the right with the contributions from all orders of $\epsilon$ combined.

For equation (2.27a), the variables $\tilde{t}, \tilde{x}$, and $\xi$ are not really independent variables
because $\xi=x-t=(\tilde{x}-\tilde{t}) / \epsilon$. Since it is not possible to change from three independent variables to two independent variables, we have to maintain the formalism that $\xi$ is independent of $\tilde{t}$ and $\tilde{x}$. The correct change of variables requires us to consider two separate sets of variable changes: first the change from $\tilde{x}$ and $\tilde{t}$ to $x$ and $t$, then the change from $\xi=x-t$ and $\eta=x+t$ to $x$ and $t$. When we perform these changes, (2.27a) becomes

$$
\begin{equation*}
F_{t}+F_{x}+\frac{\epsilon}{2} F\left(F_{x}-F_{t}\right)+\mathcal{O}\left(\epsilon^{2}\right)=0 \tag{2.28}
\end{equation*}
$$

where $F(x, t)=f(\xi, \tilde{x}, \tilde{t})$.
Because (2.27b) only contains two independent variables, the correct change of variables involves the relationships

$$
\begin{array}{ll}
\tilde{t}=\epsilon t & t=\tilde{t} / \epsilon \\
\xi=x-t & x=\xi+\tilde{t} / \epsilon
\end{array}
$$

After some algebra, (2.27b) becomes

$$
\begin{equation*}
G_{t}+G_{x}+\epsilon G G_{x}+\mathcal{O}\left(\epsilon^{2}\right)=0 \tag{2.29}
\end{equation*}
$$

where $G(x, t)=G(\xi, \tilde{x}, \tilde{t})$. Although (2.28) and (2.29) look different, once we use the fact

$$
\frac{1}{2} G_{t}+\frac{1}{2} G_{x}+\mathcal{O}(\epsilon)=0
$$

in (2.29), the two equations match up to $\mathcal{O}(\epsilon)$.
So we see that the addition of $\tilde{x}$ does not significantly affect the qualitative behavior of the outgoing wave because the governing equations with and without $\tilde{x}$ are essentially the same once we revert to physical independent variables. The effect of adding $\tilde{x}$, therefore, can only be seen while maintaining the formalism that $\xi$ and $\eta$ are independent of $\tilde{x}$ and $\tilde{t}$. For example, when we used the method of characteristics to obtain $R^{(A)}$ in (2.12), we considered $\tilde{t}, \tilde{x}$, and $\xi$ to be three independent variables.

To summarize, the presence of boundaries causes us to add stretched spatial scales to our the multiple-scale solution so that we have enough degrees of freedom to satisfy all of the initial and boundary conditions. These stretched spatial scales don't significantly affect the qualitative behavior of the solution, and their benefit is only achieved by solving consistency conditions under the assumption that all scales are independent of one another.

## Chapter 3

## PROBLEM DESCRIPTION

In this chapter, we present a standard form for analyzing systems of conservation laws with rapidly fluctuating quantities. We begin with a general system of conservation laws and expand about its steady state. This derivation is described in detail in [19], so we only give the main results here. We also discuss the boundedness of solutions to a linear problem and how initial conditions must be carefully chosen to avoid solutions that depend on $t^{*}=t / \epsilon$.

### 3.1 Perturbation of a system of conservation laws about a steady state

Consider a general system of conservation laws written in divergence form,

$$
\begin{equation*}
\mathbf{p}_{t}+\mathbf{q}_{x}=\mathbf{s}, \tag{3.1}
\end{equation*}
$$

where the conserved quantity $\mathbf{p}$, the flux $\mathbf{q}$, and the source $\mathbf{s}$ are all vectors with $n$ components. Let $\epsilon$ be a small, positive parameter: $0<\epsilon \ll 1$. Each $p_{i}, q_{i}$ and $s_{i}$ is a function of the $n$ dependent variables, $w_{1}, w_{2}, \ldots, w_{n}$, the fast spatial variable $x^{*}=x / \epsilon$, and the small parameter $\epsilon$. In other words,

$$
p_{i}=p_{i}\left(w_{1}, w_{2}, \ldots, w_{n}, x^{*} ; \epsilon\right)
$$

We assume that all the dependent variables are themselves functions of $x$ and $t$ along with any other stretched scales ( $x^{*}, \tilde{x}=\epsilon x$, or $\tilde{t}=\epsilon t$ ) that we desire. Furthermore, the dependent variables should be continuous functions or else the general system of conservation laws (3.1) must be written in its integral form.

If we assume that all $p_{i}, q_{i}$ and $s_{i}$ are $\mathcal{O}(1)$ as $\epsilon \rightarrow 0$ and that the dependence of the flux on the fast spatial variable is weak,

$$
\mathbf{q}_{x^{*}}=\epsilon \mathbf{m}\left(w_{1}, \ldots, w_{n}, x^{*} ; \epsilon\right)=\mathcal{O}(\epsilon)
$$

then the dependent variables are governed by the system of equations

$$
\begin{equation*}
\mathbf{w}_{t}+R\left(\mathbf{w}, x^{*} ; \epsilon\right) \mathbf{w}_{x}=\mathbf{r}\left(\mathbf{w}, x^{*} ; \epsilon\right) \tag{3.2}
\end{equation*}
$$

where $R=P^{-1} Q, \mathbf{r}=P^{-1}(\mathbf{s}-\mathbf{m})$, and $P$ and $Q$ are the Jacobian matrices

$$
P=\frac{\partial\left(p_{1}, \ldots, p_{n}\right)}{\partial\left(w_{1}, \ldots, w_{n}\right)}, \quad \text { and } \quad Q=\frac{\partial\left(q_{1}, \ldots, q_{n}\right)}{\partial\left(w_{1}, \ldots, w_{n}\right)}
$$

The steady state solution to (3.2), which we denote by $\mathbf{v}\left(x^{*} ; \epsilon\right)$, is governed by the quasilinear system of ordinary differential equations

$$
R\left(\mathbf{v}, x^{*} ; \epsilon\right) \mathbf{v}^{\prime}=\epsilon \mathbf{r}\left(\mathbf{v}, x^{*} ; \epsilon\right)
$$

and can always be obtained by a regular perturbation expansion [19]. Let $\mathbf{u}$ be a small perturbation to this steady state. We substitute $\mathbf{w}=\mathbf{v}+\epsilon \mathbf{u}$ into (3.2) and, after much algebra, obtain the governing equation for $\mathbf{u}$,

$$
\begin{equation*}
\mathbf{u}_{t}+A\left(x^{*}\right) \mathbf{u}_{x}+B\left(x^{*}\right) \mathbf{u}=\epsilon\left[C\left(\mathbf{u}, x^{*}\right) \mathbf{u}+D\left(\mathbf{u}, x^{*}\right) \mathbf{u}_{x}\right]+\mathcal{O}\left(\epsilon^{2}\right) . \tag{3.3}
\end{equation*}
$$

Assuming that the steady state can be expanded as

$$
\mathbf{v}\left(x^{*} ; \epsilon\right)=\mathbf{v}^{(0)}\left(x^{*}\right)+\epsilon \mathbf{v}^{(1)}\left(x^{*}\right)+\epsilon^{2} \mathbf{v}^{(2)}\left(x^{*}\right)+\mathcal{O}\left(\epsilon^{3}\right)
$$

the components of each matrix in (3.3) are

$$
\begin{aligned}
a_{i j}\left(x^{*}\right)= & r_{i j}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right) \\
b_{i j}\left(x^{*}\right)= & -\frac{\partial r_{i}}{\partial w_{j}}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right)+\frac{\partial r_{i m}}{\partial w_{j}}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right) \frac{d v_{m}^{(1)}}{d x^{*}} \\
c_{i j}\left(\mathbf{u}, x^{*}\right)= & \frac{\partial^{2} r_{i}}{\partial w_{j} \partial \epsilon}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right)+\frac{\partial^{2} r_{i}}{\partial w_{j} \partial w_{m}}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right)\left(v_{m}^{(1)}\left(x^{*}\right)+\frac{1}{2} u_{m}\right) \\
& +\frac{\partial r_{i m}}{\partial w_{j}}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right) \frac{d v_{m}^{(2)}}{d x^{*}} \\
d_{i j}\left(\mathbf{u}, x^{*}\right)= & -\frac{\partial r_{i j}}{\partial \epsilon}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right)-\frac{\partial r_{i j}}{\partial w_{m}}\left(\mathbf{v}^{(0)}\left(x^{*}\right), x^{*} ; 0\right)\left(v_{m}^{(1)}\left(x^{*}\right)+u_{m}\right),
\end{aligned}
$$

where we have adopted the summation convention over repeated indices on $m$.
We will use the standard form (3.3) repeatedly throughout this paper, as it gives us a unified way of analyzing systems of hyperbolic conservation laws. In particular, we only consider pairs of conservation laws, so all of the matrix coefficients in (3.3) are $2 \times 2$ matrices. We do not consider systems of three or more conservation laws to avoid the possibility of resonant interactions between the dependent variables for certain periodic initial conditions [17].

Furthermore, all of our examples will involve periodic $x^{*}$-fluctuations in (3.3). There is nothing that prevents us from assuming any other type of fluctuations-the only requirement is that the fluctuating functions are well-behaved so that the averaging operators defined in Appendix A can be applied to them. Although choosing periodic fluctuations makes the algebra a little simpler, it allows for the possibility of resonant interactions between waves traveling through the medium and the medium itself. We give an example of such a situation in the next section.

### 3.2 The linear problem

Since the nonlinear terms in (3.3) are premultiplied by $\epsilon$, it is the linear problem

$$
\begin{equation*}
\mathbf{u}_{t}+A\left(x^{*}\right) \mathbf{u}_{x}+B\left(x^{*}\right) \mathbf{u}=0 \tag{3.4}
\end{equation*}
$$

that governs the basic behavior of its solutions. That is why it is important to first look carefully at (3.4) before studying the general nonlinear problem.

Since the eigenvalues of $A\left(x^{*}\right)$ are the speeds at which information propagates through the system, for (3.4) to describe a hyperbolic system of partial differential equations, the matrix $A\left(x^{*}\right)$ must have real and distinct eigenvalue for all $x^{*}$. (Hyperbolic problems with distinct eigenvalues are sometimes called "strictly hyperbolic.")

Let's also assume that one of the eigenvalues is always positive and one is always negative. The effect of this assumption is to turn a boundary like $x=0$ into a space-
like arc, a contour in the $x-t$ plane that only has one characteristic emanating from it in positive time. Therefore, a well-posed boundary-value problem can only have one condition on one linear combination of the dependent variables at $x=0$. In contrast, the solution boundary $t=0$ is a time-like arc because it has two characteristics emanating from it in positive time. The reason for making this assumption about the eigenvalues will become clearer when we highlight the missing boundary condition difficulty in Chapter 4. We will see that if two boundary conditions can be specified at $x=0$, there is no missing boundary condition difficulty. If no boundary conditions can be specified at $x=0$ (for example, if both eigenvalues of $A$ are negative), then we are effectively solving an initial-value problem since boundary conditions never affect the solution in its solution domain. For convenience, we assume that the boundary conditions are always specified at $x=0$, something that can always be achieved through an appropriate change of independent variables.

### 3.3 Conditions for bounded solutions of a special linear case

We now turn to a special linear case,

$$
\begin{equation*}
\mathbf{u}_{t}+A\left(x^{*}\right) \mathbf{u}_{x}=0 \tag{3.5}
\end{equation*}
$$

with

$$
A^{-1}\left(x^{*}\right)=\left[\begin{array}{ll}
1+\alpha \sin \pi x^{*} & 2+\beta \cos \pi x^{*}  \tag{3.6}\\
2+\gamma \cos \pi x^{*} & 1+\delta \sin \pi x^{*}
\end{array}\right]
$$

where $\alpha, \beta, \gamma$, and $\delta$ are all real constants. This problem was first suggested by Kevorkian and Bosley in [19] as an example in which the cumulative effect of the fluctuations in $A\left(x^{*}\right)$ is "negative diffusion" (solution profile steepens instead of flattens). In this section we establish some preliminary facts about this problem as we intend to make use of it in later chapters. We will show that the presence of diffusion in the homogenized (averaged) behavior of (3.5) implies exponential growth of the solution.

Unlike the initial-value problem for the scalar equation $u_{t}+a\left(x^{*}\right) u_{x}=0$, or the constant coefficient matrix equation $\mathbf{u}_{t}+A \mathbf{u}_{x}=0$, whose solutions never grow larger than their initial conditions, equation (3.5) with periodic $A\left(x^{*}\right)$ matrices can have solutions that grow exponentially in time. In this section, we will try to further analyze the mechanism that causes solutions to grow exponentially.

Because (3.5) is a linear problem, we can use the principle of superposition, which reduces the task of describing how an arbitrary initial condition propagates in time down to an examination of how each individual trigonometric mode evolves. Superposition also ensures that all of the periodic modes initially present do not directly interact with each other; instead, we will see that the primary feature of (3.5) is the interaction of waves with periodic medium represented by $A\left(x^{*}\right)$. The functions in $A\left(x^{*}\right)$ have a period of $2 / \epsilon$ and wavenumber $\pi / \epsilon$. The product of $A\left(x^{*}\right)$ and $\mathbf{u}_{x}$ in (3.5) causes a wave with wavenumber $\pi m$ to excite all waves with wavenumber $\pi m(1+i / \epsilon)$ for any integer $i$. A complete analysis of the growth and decay of these modes would therefore involve studying an infinite system of ordinary differential equations governing the amplitudes of these modes.

As a starting point to understanding the behavior of (3.5), we follow [19] by supposing that the solution is of the form

$$
\mathbf{u}=\left[\begin{array}{l}
v_{1}(t) \sin (m \pi x)+v_{2}(t) \cos (m \pi x)+v_{5}(t) \sin (n \pi x)+v_{6}(t) \cos (n \pi x)  \tag{3.7}\\
v_{3}(t) \sin (m \pi x)+v_{4}(t) \cos (m \pi x)+v_{7}(t) \sin (n \pi x)+v_{8}(t) \cos (n \pi x)
\end{array}\right]
$$

where $n=1 / \epsilon-m$. When we plug ansatz (3.7) into (3.5) and ignore all other modes that are generated except for those with wavenumbers $m \pi$ and $n \pi$, we obtain the system of ordinary differential equations $d \mathbf{v} / d t=M \mathbf{v}$, where $\mathbf{v}=\left[v_{1}(t) \cdots v_{8}(t)\right]^{T}$ is
the vector of amplitudes and

$$
M^{-1}=\frac{1}{\pi}\left[\begin{array}{cccccccc}
0 & -\frac{1}{m} & 0 & -\frac{2}{m} & -\frac{\alpha}{2 m} & 0 & 0 & -\frac{\beta}{2 m}  \tag{3.8}\\
\frac{1}{m} & 0 & \frac{2}{m} & 0 & 0 & \frac{\alpha}{2 m} & -\frac{\beta}{2 m} & 0 \\
0 & -\frac{2}{m} & 0 & -\frac{1}{m} & 0 & -\frac{\gamma}{2 m} & -\frac{\delta}{2 m} & 0 \\
\frac{2}{m} & 0 & \frac{1}{m} & 0 & -\frac{\gamma}{2 m} & 0 & 0 & \frac{\delta}{2 m} \\
-\frac{\alpha}{2 n} & 0 & 0 & -\frac{\beta}{2 n} & 0 & -\frac{1}{n} & 0 & -\frac{2}{n} \\
0 & \frac{\alpha}{2 n} & -\frac{\beta}{2 n} & 0 & \frac{1}{n} & 0 & \frac{2}{n} & 0 \\
0 & -\frac{\gamma}{2 n} & -\frac{\delta}{2 n} & 0 & 0 & -\frac{2}{n} & 0 & -\frac{1}{n} \\
-\frac{\gamma}{2 n} & 0 & 0 & \frac{\delta}{2 n} & \frac{2}{n} & 0 & \frac{1}{n} & 0
\end{array}\right] .
$$

The solution (3.7) will remain bounded if and only if none of the eigenvalues of $M$ have positive real part, or equivalently, if none of the eigenvalues of $\pi M^{-1}$ have positive real part. The Routh-Hurwitz conditions, used to discriminate the roots of a polynomial, can give us some insight to the properties of these eigenvalues. The characteristic polynomial of $\pi M^{-1}$ is $\rho(\lambda)=\operatorname{det}\left(\lambda I-\pi M^{-1}\right)=\lambda^{8}+a_{7} \lambda^{7}+\cdots+$ $a_{1} \lambda+a_{0}$. We don't display the constants $a_{i}$ to save space. All of the eigenvalues of $\pi M^{-1}$, which are the roots of $\rho(\lambda)$, have negative real part if and only if all of the Hurwitz determinants are positive (see [22] and [4]). The Hurwitz determinants are the determinants of the eight principle submatrices of the matrix

$$
\left[\begin{array}{cccccccc}
a_{7} & 1 & 0 & 0 & 0 & 0 & 0 & 0  \tag{3.9}\\
a_{5} & a_{6} & a_{7} & 0 & 0 & 0 & 0 & 0 \\
a_{3} & a_{4} & a_{5} & a_{6} & 0 & 0 & 0 & 0 \\
a_{1} & a_{2} & a_{3} & a_{4} & a_{5} & 0 & 0 & 0 \\
0 & a_{0} & a_{1} & a_{2} & a_{3} & a_{4} & 0 & 0 \\
0 & 0 & 0 & a_{0} & a_{1} & a_{2} & a_{3} & 0 \\
0 & 0 & 0 & 0 & 0 & a_{0} & a_{1} & a_{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{0}
\end{array}\right] .
$$

It so happens that the third Hurwitz determinant (the determinant of the submatrix of (3.9) using the first three rows and columns) is

$$
-\frac{(\alpha-\delta)^{2}(\beta-\gamma)^{2}(m-n)^{2}}{m^{4} n^{4}}
$$

Because this Hurwitz determinant is nonpositive, a necessary condition for bounded solutions (3.7) is that $(\alpha-\delta)(\beta-\gamma)=0$. Unfortunately, it turns out that once this condition is satisfied, all the Hurwitz determinants vanish and nothing else can be said about the real parts of the eigenvalues of $-\pi M^{-1}$; that is why the Routh-Hurwitz analysis produces $(\alpha-\delta)(\beta-\gamma)=0$ only as a necessary condition.

The quantity $(\alpha-\delta)(\beta-\gamma)$ is also directly related to the diffusion coefficient for the homogenized behavior of (3.5). Using multiple-scale analysis, one can show that the cumulative effects of the fluctuations in $A\left(x^{*}\right)$ are described by the equations

$$
\begin{align*}
& \frac{\partial w_{1}^{(0)}}{\partial \tilde{t}}=\frac{(\alpha-\delta)(\beta-\gamma)}{108 \pi} \frac{\partial^{2} w_{1}^{(0)}}{\partial \xi^{2}}  \tag{3.10a}\\
& \frac{\partial w_{2}^{(0)}}{\partial \tilde{t}}=\frac{3(\alpha-\delta)(\beta-\gamma)}{4 \pi} \frac{\partial^{2} w_{2}^{(0)}}{\partial \eta^{2}} \tag{3.10b}
\end{align*}
$$

In this pair of equations, $w_{1}^{(0)}$ and $w_{2}^{(0)}$ represent the leading-order homogenized behavior of (3.5), written in characteristic independent and dependent variables. The most important feature of equations (3.10a) and (3.10b) is that they are linear heat equations and the sign of $(\alpha-\delta)(\beta-\gamma)$ determines whether $w_{1}^{(0)}$ and $w_{2}^{(0)}$ diffuse positively (spread outwards) or negatively (ever-increasing gradients). In particular, when $(\alpha-\delta)(\beta-\gamma)$ is negative these equations are ill-posed, since waves with different frequencies grow exponentially with different rates. It is therefore not surprising that the quantity $(\alpha-\delta)(\beta-\gamma)$ is related to the boundedness of solutions to (3.5).

All of this information is backed up by the following informal numerical experiment. Choose $\epsilon=0.1$, then repeat the following sequence of steps as many times as desired.

1. Randomly choose the four parameters $\alpha, \beta, \gamma$, and $\delta$ from the interval $[-1,1]$ for the matrix $A\left(x^{*}\right)$ in (3.6).
2. Generate the matrix $\pi M^{-1}$ and calculate its eigenvalues for a suitable range of $m$. (If you imagine solving (3.5) on a periodic domain, then one only needs to check the values of $m$ representing waves that satisfy the periodic boundary conditions.) Divide the eigenvalues into $\pi$ to get the eigenvalues of $M$.
3. Out of all of the eigenvalues of $M$, select the one whose real part is the most positive and one whose real part is the most negative.
4. Plot the value of $(\alpha-\delta)(\beta-\gamma)$ along with both extreme real parts.

In Figure 3.1, we have plotted 1,000 min/max pairs for 1,000 randomly chosen sets of the four parameters $\alpha, \beta, \gamma$, and $\delta$. The eigenvalues with positive real parts are plotted above the dotted line, the eigenvalues with negative real parts below. We find that the real parts of the eigenvalues vanish exactly when the quantity $(\alpha-\delta)(\beta-\gamma)$ also vanishes. The figure suggests that $(\alpha-\delta)(\beta-\gamma)=0$ is both a necessary and sufficient condition for all the eigenvalues of $M$ to be purely imaginary—a slightly stronger statement than we could obtain from the Routh-Hurwitz conditions.

Let us keep in mind that all of this analysis on the system of ordinary differential equations $\mathbf{v}^{\prime}=M \mathbf{v}$ is only a first approximation to the full behavior of (3.5) because solutions of the form (3.7) don't satisfy the partial differential equation (3.5) exactly.

Using a spectral numerical technique (see Appendix B), we numerically solve (3.5) with $A\left(x^{*}\right)$ defined by (3.6) in the domain $-1<x<1$, using periodic boundary conditions. We chose $\epsilon=0.1$. Figure 3.2 displays the results of these numerical simulations for four different choices of the four constants $\alpha, \beta, \gamma$, and $\delta$ at $t=2000$. The initial conditions for all four cases are

$$
\mathbf{u}(x, 0)=\left[\begin{array}{c}
\sin (\pi x) \\
0
\end{array}\right]
$$

which is shown in pane (a). In panes (b)-(d), we choose three different sets of


Figure 3.1: Informal numerical experiment to verify the condition $(\alpha-\delta)(\beta-\gamma)=0$.
parameters with $\alpha=\delta$ or $\beta=\gamma$ or both. Changes in these four parameters affect the solution greatly at $t=2000$, but the solution remains bounded between $\pm 1$. In pane (e), we show the numerical solution for a case in which $(\alpha-\delta)(\beta-\gamma) \neq 0$. Notice from the vertical scales that the solution grows exponentially in this case. To summarize, we observe that the numerical solution stays bounded only when $\alpha=\delta$ or $\beta=\gamma$. Even though this anecdotal evidence falls short of a proof, it suggests that we should restrict ourselves to instances of (3.6) in which $(\alpha-\delta)(\beta-\gamma)=0$.

We repeated the numerical simulation of the exponentially growing case ( $\alpha=0$, $\delta=.4, \beta=0.1, \gamma=-0.2$ ) using CLAWPACK with a spatial step size of 0.0005 . Figure 3.3 shows this numerically calculated solution at $t=200$.

(a) Initial conditions for all cases

(b) $\alpha=\delta=.5, \beta=\gamma=-0.1$

(c) $\alpha=\delta=.5, \beta=0.5, \gamma=-0.4$
(d) $\alpha=.5, \delta=0, \beta=\gamma=0.5$
(e) $\alpha=0, \delta=.4, \beta=0.1, \gamma=-0.2$

Figure 3.2: Initial condition and numerical solution to the test problem of Kevorkian and Bosley with four different sets of parameters at $t=2000$ using a spectral technique.


Figure 3.3: Numerical solution to the test problem of Kevorkian and Bosley at $t=200$ using CLAWPACK.

There is an interesting connection between Figures 3.3 and 3.2(e), which represent the same unstable choice of $\alpha, \beta, \gamma, \delta$. Ignoring the fact the two figures depict solutions at different times, notice that the solution calculated by CLAWPACK is overwhelmed by a wave with wavenumber $7 \pi$ while the solution calculated by the spectral technique shows that the wave with wavenumber $\pi$ dominates. Why is there a discrepancy between these two graphs, and which is correct?

The answer begins with the fact that the real parts of the eigenvalues of the matrix $M$, whose inverse is defined by (3.8), are most negative when $m=7$. Even though the initial condition does not contain any waves with wavenumber $7 \pi$, because of small truncation errors, a finite volume or finite difference method will always introduce waves of every wavenumber. In our simulation, these small errors are magnified until they overwhelm the solution. Increasing the spatial resolution delays, but does not prevent, exponential growth of the solution [19].

In contrast, our spectral numerical technique works by solving a system of ordinary differential equations for all the modes in the solution. If the amplitude of the mode with wavenumber $7 \pi$ is initially zero and if this wave is not excited by any other waves, then the amplitude of this wave will remain zero for all time. The $A\left(x^{*}\right)$ matrix defined in (3.6) has the feature that it contains only periodic functions with wavenumber $\pi / \epsilon$. As a result, $A\left(x^{*}\right)$ couples all modes in the solution whose wavenumbers are apart by an exact multiple of $\pi / \epsilon$. In other words, if $\epsilon=.1$, then all the modes with wavenumbers $\ldots,-13 \pi,-3 \pi, 7 \pi, 17 \pi, 27 \pi, \ldots$ are coupled. None of these modes are present in the initial conditions, so none of them will appear in the solution.

These numerical experiments suggest two reasons why we should only study (3.5) with $A\left(x^{*}\right)$ defined in (3.6) when $(\alpha-\delta)(\beta-\gamma)=0$. First, to compare our analytic solutions with numerical results we need reliable numerical results-results that don't significantly change because of changes in the step-size, $\Delta x$. Second, problems in which waves of different wavenumbers grow exponentially at different rates are inherently ill-conditioned. In other words, we are not guaranteed that two solutions corresponding to two initial conditions that are close to one another will remain close to one another as they evolve in time.

To summarize the results in this section, we have proved that for the example (3.6) devised by Kevorkian and Bosley, nonzero $(\alpha-\delta)(\beta-\gamma)$ leads to solutions that grow exponentially in time. These solutions grow unbounded because of resonant interactions between waves traveling through the medium and medium itself. Furthermore, we have numerical evidence to suggest that $(\alpha-\delta)(\beta-\gamma)=0$ is necessary and sufficient for bounded solutions. Although Kevorkian and Bosley showed that diffusion (even "negative" diffusion) is a real effect that must be accounted for when $(\alpha-\delta)(\beta-\gamma)$ is nonzero, they did not realize that such solutions would grow exponentially. From this point on, we will assume $(\alpha-\delta)(\beta-\gamma)=0$ whenever we refer to this example.

### 3.4 Choosing the initial conditions for a $t^{*}$-independent solution

In preparation for the analysis in the next chapter, we now show how to choose initial conditions for (3.3) so that the solution is independent of $t^{*}=t / \epsilon$.

Why is it important to have solutions that are independent of $t^{*}$ ? If the functions represented by $A, B, C$ and $D$ in (3.3) depend on $x$, the natural spatial scale of the problem, a closed-form solution to (3.3) would be out of reach. It is the assumption that these functions depend on $x^{*}$ that allows us to make some progress through multiple-scale analysis. Likewise, if the solution to (3.3) depends on $t^{*}$, then it is possible to use a simple change of variables to rescale the problem so that the natural fluctuations are on the same scale as the fluctuations of the coefficient matrices. Therefore, it is important to ensure that solutions to (3.3) are independent of the fast temporal scale $t^{*}=t / \epsilon$.

In [30], Santosa and Symes tacitly assume that their solution is independent of the fast temporal scale, although not in the context of multiple-scale analysis. Their analysis of the linear wave equation

$$
\begin{equation*}
\rho\left(x^{*}\right) w_{t t}-\left(k\left(x^{*}\right) w_{x}\right)_{x}=0, \tag{3.11}
\end{equation*}
$$

where $w(x, t ; \epsilon)$ is the displacement, the density and bulk modulus of the medium, $\rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$, involves a Bloch wave expansion-essentially a spectral decomposition of the partial differential operator for (3.11) that recasts the initial-value problem as an eigenvalue problem. They express their solution as

$$
w(x, t ; \epsilon)=\int_{|k| \leq 2 \pi} g(k) \exp (i k x \pm i \omega(k) t) d k+\mathcal{O}(\epsilon),
$$

where $\omega(k)$ are the eigenvalues associated with each Bloch wave. The important assumption here is that $k$, a scaled wavenumber, is restricted by the integration limits. This assumption that the solution is band-limited achieves the same result as assuming that the solution is independent of the fast temporal scale $t^{*}$.

So how does one ensure that the solution is independent of $t^{*}$ ? According to Kevorkian and Bosley, only problems with special initial conditions lead to solutions without high-frequency temporal oscillations [19]. Ironically, it is the careful choice of the $x^{*}$-dependence of the initial conditions that leads to $t^{*}$-independent solutions. Determining the necessary $x^{*}$-dependence is easy-from the calculated multiple-scale solution, one sets $t=0$ to see what initial conditions are necessary to support that solution. We will see that the averaged (or $x^{*}$-independent) behavior of the initial conditions may be chosen arbitrarily, but the $x^{*}$-dependence may not. Typically, the initial conditions will be of the form

$$
\mathbf{u}(x, 0 ; \epsilon)=\mathbf{h}(x ; \epsilon)=\underline{\mathbf{h}}^{(0)}(x)+\epsilon \mathbf{h}^{(1)}\left(x^{*}, x\right)+\cdots
$$

Notice that, as expected, the leading-order initial conditions must be independent of $x^{*}$ for us to make any progress on the solution to (3.3). The $x^{*}$-dependencies of all higher order terms in the initial conditions cannot be arbitrarily chosen. Practically speaking, this means that any solution that we obtain using multiple scales is actually the solution to a nearby problem; the smaller $\epsilon$ gets, the better our approximation becomes.

Finally, we also cite the finding that discontinuous initial data lead to solutions that are $x^{*}$ - and $t^{*}$-dependent in the region of influence of the discontinuity [19]. This finding means we should also ensure that our initial and boundary data are continuous functions and that they match up correctly at any places in the solution domain where they meet. Furthermore, problems whose solutions exhibit shocks require the use of $x^{*}$ and $t^{*}$ after the time of shock formation.

## Chapter 4

## MULTIPLE-SCALE ANALYSIS

In this chapter, we apply multiple-scale analysis to problems in the standard form (3.3). We begin by analyzing a linear problem to demonstrate the multiple-scale method, the difficulty of the missing boundary conditions, and how to overcome it. The ideas from this linear problem are then applied to more general nonlinear systems of conservation laws.

### 4.1 The linear problem with $B=0$

Before we analyze the general nonlinear problem, we begin by studying the the linear problem

$$
\begin{equation*}
\mathbf{u}_{t}+A\left(x^{*}\right) \mathbf{u}_{x}=\mathbf{0} \tag{4.1}
\end{equation*}
$$

which corresponds to the linear part of (3.3) with $B\left(x^{*}\right)=0$. This equation is insightful because it demonstrates all of the main ideas of this chapter. We consider two problems fitting this mathematical description: the linear wave equation with rapidly varying density and bulk modulus, and a nonphysical test case devised by Kevorkian and Bosley [19].

Written as a second-order partial differential equation, the linear wave equation is

$$
\begin{equation*}
\rho\left(x^{*}\right) w_{t t}-\left(k\left(x^{*}\right) w_{x}\right)_{x}=0, \tag{4.2}
\end{equation*}
$$

where $w(x, t ; \epsilon)$ is the displacement. Here, we allow the density and bulk modulus of the medium, $\rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$, to vary rapidly on the $x^{*}=x / \epsilon$ scale. Changing
variables using $u_{1}=w_{t}$ and $u_{2}=-k w_{x}$ puts (4.2) in the form of (4.1) with

$$
A_{\text {waves }}\left(x^{*}\right)=\left[\begin{array}{cc}
0 & 1 / \rho\left(x^{*}\right)  \tag{4.3}\\
k\left(x^{*}\right) & 0
\end{array}\right] .
$$

The description of linear (small amplitude) acoustic waves in a stationary gas is one of many applications of the linear wave equation. In that context, $u_{1}$ and $u_{2}$ stand for the velocity and pressure, respectively, and the speed of sound is $c\left(x^{*}\right)=\sqrt{k\left(x^{*}\right) / \rho\left(x^{*}\right)}$.

In [19], Kevorkian and Bosley examine (4.1) with

$$
A_{\mathrm{test}}^{-1}\left(x^{*}\right)=\left[\begin{array}{ll}
1+\alpha \sin \left(\pi x^{*}\right) & 2+\beta \cos \left(\pi x^{*}\right) \\
2+\gamma \cos \left(\pi x^{*}\right) & 1+\delta \sin \left(\pi x^{*}\right)
\end{array}\right],
$$

where $\alpha, \beta, \gamma$ and $\delta$ are real constants. Following the discussion in Section 3.3, we choose $(\alpha-\delta)(\beta-\gamma)=0$ to avoid solutions that grow exponentially. For both the linear wave equation and the test problem, we will see that the primary effect of the fluctuations in $A\left(x^{*}\right)$ is the addition of a small amount of dispersion to the solution dynamics.

The parameters in both problems must be chosen so that they describe hyperbolic problems (the eigenvalues must be real and distinct). Furthermore, as discussed in Section 3.2, we want to pick $k\left(x^{*}\right), \rho\left(x^{*}\right), \alpha, \beta, \gamma$, and $\delta$ so that one of the eigenvalues is always positive and the other always negative. For the linear wave equation, as long as $k\left(x^{*}\right)$ and $\rho\left(x^{*}\right)$ are strictly positive functions, this will be true. There are no simple conditions to ensure that the eigenvalues of $A_{\text {test }}$ follow this pattern-the easiest thing to do is to pick specific values and check the eigenvalues directly.

Our solution procedure consists of

1. expanding $\mathbf{u}$ into an asymptotic series containing all of the spatial and temporal scales that we will use throughout the problem,
2. using averaging operators to separate terms that depend on $x^{*}$ from the terms that do not,
3. removing potentially secular terms from the $x^{*}$-homogenized (averaged) equations to produce long-term evolution equations (consistency conditions),
4. and solving the long-term evolution equations, using recovered boundary information, if necessary.

This analysis results in a qualitative understanding of the long-term behavior and explicit expressions for the averaged (homogenized) solution.

This method differs slightly from that outlined by Kevorkian and Bosley in [19], in which their asymptotic expansion for $\mathbf{u}$ only incorporates $x^{*}, x$, and $t$. The resulting homogenized equations tell us the cumulative effects of the fluctuations in $A\left(x^{*}\right)$, but to actually solve the homogenized equations one must expand the solution with the desired slow temporal and stretched spatial scales. We merely incorporate both expansions from the beginning.

Choosing the correct scales to include in our multiple-scale analysis is the key to the whole problem. As the goal of homogenization theory is to characterize the macroscopic behavior of media with microscopic structure, we anticipate that the cumulative effects of the fluctuations of the media on the $x^{*}=x / \epsilon$ scale will have long-term effects on the solution. If we want to capture these effects, we need to include some combination of slow temporal scales and stretched spatial scales into our multiple-scale analysis.

Whether to use slow temporal scales, stretched spatial scales, or both, is largely determined through trial and error-if there are not enough degrees of freedom to satisfy all of the initial and boundary conditions, add more scales. One general guideline is that initial value problems typically require slow temporal scales only. The presence of boundaries may require us to add stretched spatial scales, as we saw in the wavemaker problem in Section 2.1.

Precisely how slow or stretched our scales should be is largely determined by the
type of behavior we wish to capture. The long-term evolution equation arises from the removal of potentially secular terms and it tells us how the solution depends on the slow or stretched scales. Therefore, including different slow and stretched scales leads to different long-term evolution equations. Later, we will see that if the cumulative effect of the fluctuations of the media is diffusion, then terms with two derivatives in $t$ will appear in the $x^{*}$-homogenized equations at $\mathcal{O}(\epsilon)$; the $\tilde{t}=\epsilon t$ or $\tilde{x}=\epsilon x$ scale must be used to capture these behaviors. A cumulative dispersive effect, characterized by three derivatives in $t$ and/or $x$, shows up at the $\mathcal{O}\left(\epsilon^{2}\right)$ system of $x^{*}$-homogenized equations, and one must include either the slow temporal scale $\hat{t}=\epsilon^{2} t$ or the stretched spatial scale $\hat{x}=\epsilon^{2} x$ to see this effect. In general, using $\epsilon^{n} x$ or $\epsilon^{n} t$ enables us to obtain a long-term evolution equation from the $\mathcal{O}\left(\epsilon^{n}\right)$ system of $x^{*}$-homogenized equations.

In this problem, we wish to take into account the cumulative dispersive effects of the fluctuations in $A\left(x^{*}\right)$, so we use the asymptotic expansion

$$
\mathbf{u}(x, t ; \epsilon)=\mathbf{u}^{(0)}\left(x^{*}, x, t, \hat{t}\right)+\epsilon \mathbf{u}^{(1)}\left(x^{*}, x, t, \hat{t}\right)+\cdots .
$$

We don't include any stretched spatial scales because they aren't necessary in a linear problem like (4.1). With this choice of scales, derivatives with respect to $x$ and $t$ become

$$
\begin{aligned}
\frac{\partial}{\partial x} & \rightarrow \epsilon^{-1} \frac{\partial}{\partial x^{*}}+\frac{\partial}{\partial x} \\
\frac{\partial}{\partial t} & \rightarrow \frac{\partial}{\partial t}+\epsilon^{2} \frac{\partial}{\partial \hat{t}}
\end{aligned}
$$

(This change of variables is actually an abuse of notation-to eliminate confusion, one should really use $x$ to stand for the original spatial scale, and some other variable like $y=x$ to stand for the same scale after the other spatial scales are introduced.) We plug

$$
\begin{aligned}
& \mathbf{u}_{t} \rightarrow \mathbf{u}_{t}^{(0)}+\epsilon \mathbf{u}_{t}^{(1)}+\epsilon^{2}\left(\mathbf{u}_{\hat{t}}^{(0)}+\mathbf{u}_{t}^{(2)}\right)+\cdots \quad \text { and } \\
& \mathbf{u}_{x} \rightarrow \epsilon^{-1} \mathbf{u}_{x^{*}}^{(0)}+\left(\mathbf{u}_{x}^{(0)}+\mathbf{u}_{x^{*}}^{(1)}\right)+\epsilon\left(\mathbf{u}_{x}^{(1)}+\mathbf{u}_{x^{*}}^{(2)}\right)+\epsilon^{2}\left(\mathbf{u}_{x}^{(2)}+\mathbf{u}_{x^{*}}^{(3)}\right)+\cdots .
\end{aligned}
$$

into the original equation (4.1) and separate terms according to their associated power of $\epsilon$.

### 4.1.1 $\mathcal{O}\left(\epsilon^{-1}\right)$ system

When we collect all terms that are proportional to $\epsilon^{-1}$, the system of equations

$$
A\left(x^{*}\right) \mathbf{u}_{x^{*}}^{(0)}=\mathbf{0}
$$

arises. Since the eigenvalues of $A\left(x^{*}\right)$ are never zero, $A\left(x^{*}\right)$ may be inverted to obtain $\mathbf{u}_{x^{*}}^{(0)}=\mathbf{0}$. In other words, the leading-order behavior of the solution does not depend on the fast spatial scale. We follow the convention that all quantities independent of the fast spatial scale are underlined. For example, we denote $\mathbf{u}^{(0)}=\underline{\mathbf{u}}^{(0)}(x, t, \hat{t})$.

### 4.1.2 $\mathcal{O}(1)$ system

The $\mathcal{O}(1)$ system of equations,

$$
\begin{equation*}
\mathbf{u}_{x^{*}}^{(1)}=-A^{-1} \underline{\mathbf{u}}_{t}^{(0)}-\underline{\mathbf{u}}_{x}^{(0)}, \tag{4.4}
\end{equation*}
$$

may be easily integrated with respect to $x^{*}$, but before we do this we need to separate the terms that depend on the fast spatial scale from the terms that do not. When $x^{*}$-independent terms are integrated with respect to $x^{*}$, terms proportional to $x^{*}$ arise. Such terms, examples of so-called secular terms, are not allowed because they cause the asymptotic expansion to become nonuniform for large $x^{*}$. In other words, as $x^{*}$ increases, the terms in the asymptotic expansion outgrow their assigned orders of magnitude.

To separate the $x^{*}$-dependent terms from the $x^{*}$-independent terms, we use the averaging operators defined in Appendix A. Setting the average ( $x^{*}$-independent) part of the right-hand side of (4.4) to zero, we get the $x^{*}$-homogenized equation

$$
\begin{equation*}
\underline{\mathbf{u}}_{t}^{(0)}+\left\langle A^{-1}\right\rangle^{-1} \underline{\mathbf{u}}_{x}^{(0)}=\mathbf{0} . \tag{4.5}
\end{equation*}
$$

The multiple-scale analysis shows that the correct description of the average behavior of our system is obtained by replacing the fluctuating $A\left(x^{*}\right)$ with the constant matrix $\left\langle A^{-1}\right\rangle^{-1}$; the naive approach of replacing $A\left(x^{*}\right)$ with $\left\langle A\left(x^{*}\right)\right\rangle$ is incorrect. In the context of the linear wave equation,

$$
\left\langle A_{\text {waves }}^{-1}\right\rangle^{-1}=\left[\begin{array}{cc}
0 & 1 /\left\langle\rho\left(x^{*}\right)\right\rangle \\
\left\langle 1 / k\left(x^{*}\right)\right\rangle^{-1} & 0
\end{array}\right] .
$$

It is interesting that while the density is replaced by its average value, the bulk modulus must be replaced by its harmonic average to produce the correct averaged behavior.

The remaining (fluctuating) part of the $\mathcal{O}(1)$ system is

$$
\mathbf{u}_{x^{*}}^{(1)}=-\left\{A^{-1}\left(x^{*}\right)\right\} \underline{\mathbf{u}}_{t}^{(0)},
$$

which, when integrated, becomes

$$
\begin{equation*}
\mathbf{u}^{(1)}=-\llbracket A^{-1}\left(x^{*}\right) \rrbracket \underline{\mathbf{u}}_{t}^{(0)}+\underline{\mathbf{u}}^{(1)}(x, t, \hat{t}), \tag{4.6}
\end{equation*}
$$

where $\underline{\mathbf{u}}^{(1)}(x, t, \hat{t})$ is the constant of integration.

### 4.1.3 $\mathcal{O}(\epsilon)$ system

The governing equations for $\mathbf{u}^{(1)}$ are

$$
\mathbf{u}_{t}^{(1)}+A\left(x^{*}\right)\left(\mathbf{u}_{x}^{(1)}+\mathbf{u}_{x^{*}}^{(2)}\right)=\mathbf{0} .
$$

We plug in (4.6) and rearrange to get

$$
\mathbf{u}_{x^{*}}^{(2)}=-A^{-1} \underline{\mathbf{u}}_{t}^{(1)}-\underline{\mathbf{u}}_{x}^{(1)}+\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t x}^{(0)}+A^{-1} \llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t t}^{(0)} .
$$

The removal of secular terms produces the equation

$$
\begin{equation*}
\underline{\mathbf{u}}_{t}^{(1)}+\left\langle A^{-1}\right\rangle^{-1} \underline{\mathbf{u}}_{x}^{(1)}=\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \rrbracket\right\rangle \underline{\mathbf{u}}_{t t}^{(0)} . \tag{4.7}
\end{equation*}
$$

The remaining (fluctuating) part of the $\mathcal{O}(\epsilon)$ system is

$$
\mathbf{u}_{x^{*}}^{(2)}=-\left\{A^{-1}\right\} \underline{\mathbf{u}}_{t}^{(1)}+\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t x}^{(0)}+\left\{A^{-1} \llbracket A^{-1} \rrbracket\right\} \underline{\mathbf{u}}_{t t}^{(0)},
$$

which, when integrated, becomes

$$
\begin{equation*}
\mathbf{u}^{(2)}=-\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t}^{(1)}+\llbracket \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t x}^{(0)}+\llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t t}^{(0)}+\underline{\mathbf{u}}^{(2)}(x, t, \hat{t}) \tag{4.8}
\end{equation*}
$$

where $\underline{\mathbf{u}}^{(2)}(x, t, \hat{t})$ is the constant of integration.

### 4.1.4 $\mathcal{O}\left(\epsilon^{2}\right)$ system

Plug in (4.6) and (4.8) into the $\mathcal{O}\left(\epsilon^{2}\right)$ system,

$$
\underline{\mathbf{u}}_{\hat{t}}^{(0)}+\mathbf{u}_{t}^{(2)}+A\left(x^{*}\right)\left(\mathbf{u}_{x}^{(2)}+\mathbf{u}_{x^{*}}^{(3)}\right)=\mathbf{0}
$$

and rearrange to get

$$
\begin{aligned}
\mathbf{u}_{x^{*}}^{(3)}= & -\underline{\mathbf{u}}_{x}^{(2)}-A^{-1}\left(\underline{\mathbf{u}}_{\hat{t}}^{(0)}+\underline{\mathbf{u}}_{t}^{(2)}\right) \\
& +A^{-1} \llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t t}^{(1)}-A^{-1} \llbracket \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t t x}^{(0)}-A^{-1} \llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t t t}^{(0)} \\
& +\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t x}^{(1)}-\llbracket \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \underline{\mathbf{u}}_{t x x}^{(0)} \\
& -\llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t t x}^{(0)} .
\end{aligned}
$$

The removal of secular terms produces this equation governing $\underline{\mathbf{u}}^{(2)}$,

$$
\begin{align*}
\underline{\mathbf{u}}_{\hat{t}}^{(0)}+\underline{\mathbf{u}}_{t}^{(2)}+\left\langle A^{-1}\right\rangle^{-1} \underline{\mathbf{u}}_{x}^{(2)}= & \left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \rrbracket\right\rangle \underline{\mathbf{u}}_{t t}^{(1)} \\
& -\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket \llbracket A^{-1} \rrbracket \rrbracket\right\rangle \underline{\mathbf{u}}_{t t x}^{(0)}  \tag{4.9}\\
& -\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket\right\rangle \underline{\mathbf{u}}_{t t t}^{(0)} .
\end{align*}
$$

Integrate the remaining part of the $\mathcal{O}\left(\epsilon^{2}\right)$ system to get

$$
\begin{aligned}
\mathbf{u}^{(3)}= & -\llbracket A^{-1} \rrbracket\left(\underline{\mathbf{u}}_{\hat{t}}^{(0)}+\underline{\mathbf{u}}_{t}^{(2)}\right)+\llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t t}^{(1)} \\
& \left.-\llbracket A^{-1} \llbracket \llbracket A^{-1} \rrbracket \rrbracket \rrbracket\right] \underline{\mathbf{u}}_{t t x}^{(0)}-\llbracket A^{-1} \llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \rrbracket \underline{\mathbf{u}}_{t t t}^{(0)} \\
& +\llbracket \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t x}^{(1)}-\llbracket \llbracket \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \underline{\mathbf{u}}_{t x x}^{(0)} \\
& -\llbracket \llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \underline{\mathbf{u}}_{t t x}^{(0)}+\underline{\mathbf{u}}^{(3)}(x, t, \hat{t}) .
\end{aligned}
$$

This process of removing terms independent of $x^{*}$ before integrating can be repeated to as high as degree of $\epsilon$ as desired. However, we will soon see that with our current choice of scales (in particular, $\hat{t}=\epsilon^{2} t$ ), it is not necessary to proceed any further.

### 4.1.5 Choosing initial conditions for $t^{*}$-independent solutions

Now that we have expressions for $\underline{\mathbf{u}}^{(0)}, \mathbf{u}^{(1)}$, and $\mathbf{u}^{(2)}$, we can answer the question of how to choose initial conditions so that the solution to (4.1) is independent of $t^{*}=t / \epsilon$. (See discussion in Section 3.4.) We simply substitute $t=0$ into each of these expressions to see what sorts of initial conditions support our assumption that $t^{*}$ is absent from the solution.

Suppose the initial conditions have the multiple-scale expansion

$$
\mathbf{u}(x, 0 ; \epsilon)=\mathbf{h}(x ; \epsilon)=\underline{\mathbf{h}}^{(0)}(x)+\epsilon \mathbf{h}^{(1)}\left(x^{*}, x\right)+\epsilon^{2} \mathbf{h}^{(2)}\left(x^{*}, x\right)+\cdots
$$

We write the leading term in this expansion as $\underline{\mathbf{h}}^{(0)}(x)$ because $\left.\underline{\mathbf{u}}^{(0)}\right|_{t=0}=\underline{\mathbf{h}}^{(0)}$ and $\underline{\mathbf{u}}^{(0)}$ is independent of $x^{*}$.

Likewise, matching $\left.\mathbf{u}^{(1)}\right|_{t=0}=\mathbf{h}^{(1)}$ using (4.6) gives

$$
\left.\mathbf{u}^{(1)}\right|_{t=0}=-\left.\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t}^{(0)}\right|_{t=0}+\left.\underline{\mathbf{u}}^{(1)}\right|_{t=0} .
$$

To calculate $\left.\underline{\mathbf{u}}_{t}^{(0)}\right|_{t=0}$, we use (4.5) to obtain

$$
\left.\underline{\mathbf{u}}_{t}^{(0)}\right|_{t=0}=-\left.\left\langle A^{-1}\right\rangle^{-1} \underline{\mathbf{u}}_{x}^{(0)}\right|_{t=0}=-\left\langle A^{-1}\right\rangle^{-1} \frac{d \mathbf{h}^{(0)}(x)}{d x}
$$

Therefore, we must choose

$$
\mathbf{h}^{(1)}\left(x^{*}, x\right)=\llbracket A^{-1} \rrbracket\left\langle A^{-1}\right\rangle^{-1} \frac{d \mathbf{h}^{(0)}(x)}{d x}+\underline{\mathbf{h}}^{(1)}(x)
$$

where $\underline{\mathbf{h}}^{(1)}(x)$, the average part of $\mathbf{h}^{(1)}$, may be chosen arbitrarily, but the fluctuating ( $x^{*}$-dependent) part may not.

The same analysis produces this restriction on $\mathbf{h}^{(2)}\left(x, x^{*}\right)$,

$$
\begin{aligned}
\mathbf{h}^{(2)}\left(x, x^{*}\right)= & -\llbracket A^{-1} \rrbracket\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \rrbracket\right\rangle\left\langle A^{-1}\right\rangle^{-2} \frac{d^{2} \underline{\mathbf{h}}^{(0)}(x)}{d x^{2}} \\
& +\left(\llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket\left\langle A^{-1}\right\rangle^{-2}-\llbracket \llbracket A^{-1} \rrbracket \rrbracket\left\langle A^{-1}\right\rangle^{-1}\right) \frac{d^{2} \underline{\mathbf{h}}^{(0)}(x)}{d x^{2}} \\
& +\llbracket A^{-1} \rrbracket\left\langle A^{-1}\right\rangle^{-1} \frac{d \mathbf{h}^{(1)}(x)}{d x}+\underline{\mathbf{h}}^{(2)}(x),
\end{aligned}
$$

where $\underline{\mathbf{h}}^{(2)}(x)$ may be specified arbitrarily.
In practice, the average part of the higher-order initial conditions is set to zero because for all physically relevant problems, initial data can be captured in the leadingorder initial conditions, $\underline{\mathbf{h}}^{(0)}(x)$-there is no need to include corrections at higher orders of $\epsilon$, except for those necessary for $t^{*}$-independent solutions.

### 4.1.6 Solving the homogenized equations for an initial-value problem

All of the analysis done up to this point is applicable to problems with and without boundary conditions. Before we tackle the difficulties that boundaries add to this problem, we first demonstrate that our solution procedure works for the initial-value problem. We are most concerned about whether we can generate homogenized equations that accurately describe the long-term evolution of equation (4.1).

These long-term evolution equations arise while solving the $x^{*}$-homogenized equations from each order of $\epsilon$. It is only now that we begin to see how different $A\left(x^{*}\right)$ lead to different types of long-term evolution equations, and necessitate different choices of slow temporal or stretched spatial scales.

The $x^{*}$-homogenized equation from the $\mathcal{O}(1)$ system of equations, (4.5), is

$$
\underline{\mathbf{u}}_{t}^{(0)}+\left\langle A^{-1}\right\rangle^{-1} \underline{\mathbf{u}}_{x}^{(0)}=\mathbf{0} .
$$

To solve this system, diagonalize $\left\langle A^{-1}\right\rangle^{-1}=P \wedge P^{-1}$ so that

$$
\Lambda=\left[\begin{array}{cc}
\lambda_{1} & 0 \\
0 & \lambda_{2}
\end{array}\right]
$$

Without loss of generality, we choose $\lambda_{1}$ to be the positive eigenvalue, and $\lambda_{2}$ to be the negative one.

Making the substitution $\underline{\mathbf{u}}^{(i)}=P \underline{\mathbf{w}}^{(i)}$ (the $\underline{\mathbf{w}}^{(i)}$ variables are called the characteristic dependent variables) and multiplying on the left by $P^{-1}$, we get

$$
\underline{\mathbf{w}}_{t}^{(0)}+\Lambda \underline{\mathbf{w}}_{x}^{(0)}=\mathbf{0},
$$

which can be written out as

$$
\begin{aligned}
& \underline{w}_{1 t}^{(0)}+\lambda_{1} \underline{w}_{1 x}^{(0)}=0, \\
& \underline{w}_{2 t}^{(0)}+\lambda_{2} \underline{w}_{2 x}^{(0)}=0 .
\end{aligned}
$$

These equations imply that $\underline{w}_{1}^{(0)}=\underline{w}_{1}^{(0)}(\xi, \hat{t})$, and $\underline{w}_{2}^{(0)}=\underline{w}_{2}^{(0)}(\eta, \hat{t})$, where $\xi=x-\lambda_{1} t$ and $\eta=x-\lambda_{2} t$ are called the characteristic independent variables. The dependence of $\underline{\mathbf{w}}^{(0)}$ on the slow time scale, $\hat{t}$, cannot be determined until we consider the equations arising at higher orders.

We rewrite the $\mathcal{O}(\epsilon)$ system of $x^{*}$-homogenized equations (4.7) in terms of the characteristic dependent variables to get

$$
\underline{\mathbf{w}}_{t}^{(1)}+\Lambda \underline{\mathbf{w}}_{x}^{(1)}=\mathcal{N} \underline{\mathbf{w}}_{t t}^{(0)},
$$

where $\mathcal{N}=P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \rrbracket\right\rangle P$. Recalling that $\underline{w}_{1}^{(0)}$ doesn't depend on $\eta$ and $\underline{w}_{2}^{(0)}$ doesn't depend on $\xi$, and using the facts $\partial_{x} \rightarrow \partial_{\xi}+\partial_{\eta}$ and $\partial_{t} \rightarrow-\lambda_{1} \partial_{\xi}-\lambda_{2} \partial_{\eta}$, we can rewrite the equation above using characteristic independent variables:

$$
\left[\begin{array}{l}
\left(\lambda_{1}-\lambda_{2}\right) w_{1 \eta}^{(1)}  \tag{4.10}\\
\left(\lambda_{2}-\lambda_{1}\right) w_{2 \xi}^{(1)}
\end{array}\right]=\mathcal{N}\left[\begin{array}{l}
\lambda_{1}^{2} w_{1 \xi \xi}^{(0)} \\
\lambda_{2}^{2} w_{2 \eta \eta}^{(0)}
\end{array}\right] .
$$

Before we integrate to solve for $\underline{\mathbf{w}}^{(1)}$, we must remove terms independent of $\eta$ in the first component of (4.10), terms independent of $\xi$ in the second component of (4.10). These terms are premultiplied by the diagonal entries of $\mathcal{N}$.

For the linear wave equation,

$$
\mathcal{N}_{\text {waves }}=\left[\begin{array}{cc}
0 & \left\langle k^{-1}\right\rangle^{-1 / 2}\langle\rho\rangle^{-1 / 2}\left\langle k^{-1} \llbracket \rho \rrbracket\right\rangle \\
\left\langle k^{-1}\right\rangle^{-1 / 2}\langle\rho\rangle^{-1 / 2}\left\langle\rho \llbracket k^{-1} \rrbracket\right\rangle & 0
\end{array}\right],
$$

(in this calculation we have used the fact that $\left\langle\rho \llbracket k^{-1} \rrbracket\right\rangle=-\left\langle k^{-1} \llbracket \rho \rrbracket\right\rangle$, which is proved in Appendix A), and for the test case devised by Kevorkian and Bosley,

$$
\mathcal{N}_{\text {test }}=\frac{1}{12 \pi}\left[\begin{array}{cc}
(\alpha-\delta)(\beta-\gamma) & (\alpha-\delta)(\beta+\gamma) \\
3(\alpha-\delta)(\beta+\gamma) & 3(\alpha-\delta)(\beta-\gamma)
\end{array}\right] .
$$

Recalling from our discussion in Section 3.3 that $(\alpha-\delta)(\beta-\gamma)=0$, we see that in both examples the diagonal entries of $\mathcal{N}$ are zero. As there is no need to remove any potentially secular terms, we can integrate (4.10) to obtain

$$
\begin{align*}
& \underline{w}_{1}^{(1)}=\frac{n_{12}}{\lambda_{1}-\lambda_{2}} \lambda_{2}^{2} \underline{w}_{2 \eta}^{(0)}+\underline{v}_{1}^{(1)}(\xi, \hat{t}),  \tag{4.11a}\\
& \underline{w}_{2}^{(1)}=\frac{n_{21}}{\lambda_{2}-\lambda_{1}} \lambda_{1}^{2} \underline{w}_{1 \xi}^{(0)}+\underline{v}_{2}^{(1)}(\eta, \hat{t}), \tag{4.11b}
\end{align*}
$$

where $\underline{v}_{1}^{(1)}$ and $\underline{v}_{2}^{(1)}$ are integration constants.
If the diagonal entries of $\mathcal{N}$ were nonzero, we would need to eliminate some terms from (4.10) by introducing either $\tilde{t}=\epsilon t$ or $\tilde{x}=\epsilon x$ into our asymptotic expansion. This added degree of freedom would give us the ability to eliminate those potential secular terms by forcing $\underline{w}_{1}^{(0)}$ and $\underline{w}_{2}^{(0)}$ to satisfy two constant-coefficient diffusion equations. So now we see that $\mathcal{N}$ having zero diagonal entries means that $\tilde{t}=\epsilon t$ is absent from our asymptotic expansion, and that the long-term behavior of the solution does not exhibit diffusion.

Finally, we rewrite equation (4.9), the $x^{*}$-homogenized equation from the $\mathcal{O}\left(\epsilon^{2}\right)$ system, using characteristic dependent variables to obtain

$$
\underline{\mathbf{w}}_{t}^{(0)}+\underline{\mathbf{w}}_{t}^{(2)}+\Lambda \underline{\mathbf{w}}_{x}^{(2)}=\mathcal{N} \underline{\mathbf{w}}_{t t}^{(1)}-\mathcal{R} \underline{\mathbf{w}}_{t t t}^{(0)}-\mathcal{S} \underline{\mathbf{w}}_{t t x}^{(0)},
$$

where

$$
\begin{aligned}
\mathcal{R} & =P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket\right\rangle P, \\
\text { and } \quad \mathcal{S} & =P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket \llbracket A^{-1} \rrbracket \rrbracket\right\rangle P .
\end{aligned}
$$

Using (4.11), and rewriting in terms of characteristic independent variables, we arrive at

$$
\left[\begin{array}{l}
\left(\lambda_{1}-\lambda_{2}\right) w_{1 \eta}^{(2)}  \tag{4.12}\\
\left(\lambda_{2}-\lambda_{1}\right) w_{2 \xi}^{(2)}
\end{array}\right]=\mathcal{N}\left[\begin{array}{l}
\frac{n_{12}}{\lambda_{1}-\lambda_{2}} \lambda_{2}^{4} \underline{w}_{2 \eta \eta \eta}^{(0)}+\lambda_{1}^{2} \underline{v}_{1 \xi \xi}^{(1)} \\
\frac{n_{21}}{\lambda_{2}-\lambda_{1}} \lambda_{1}^{4} \underline{w}_{1 \xi \xi \xi}^{(0)}+\lambda_{2}^{2} \underline{v}_{2 \eta \eta}^{(1)}
\end{array}\right]+\mathcal{R}\left[\begin{array}{l}
\lambda_{1}^{3} w_{1 \xi \xi \xi}^{(0)} \\
\lambda_{2}^{3} w_{2 \eta \eta \eta}^{(0)}
\end{array}\right]-\mathcal{S}\left[\begin{array}{l}
\lambda_{1}^{2} w_{1 \xi \xi \xi}^{(0)} \\
\lambda_{2}^{2} w_{2 \eta \eta \eta}^{(0)}
\end{array}\right] .
$$

Before we integrate to obtain $\underline{\mathbf{w}}^{(2)}$, we need to remove all terms independent of $\eta$ from the first component of (4.12) and all terms independent of $\xi$ from the second component, since these will lead to secular terms. This removal produces the consistency conditions

$$
\begin{aligned}
& \underline{w}_{1 \hat{t}}^{(0)}=\left(\frac{n_{12} n_{21}}{\lambda_{2}-\lambda_{1}} \lambda_{1}^{4}+r_{11} \lambda_{1}^{3}-s_{11} \lambda_{1}^{2}\right) \underline{w}_{1 \xi \xi \xi}^{(0)} \\
& \underline{w}_{2 \hat{t}}^{(0)}=\left(\frac{n_{12} n_{21}}{\lambda_{1}-\lambda_{2}} \lambda_{2}^{4}+r_{22} \lambda_{2}^{3}-s_{22} \lambda_{2}^{2}\right) \underline{w}_{2 \eta \eta \eta}^{(0)}
\end{aligned}
$$

To solve these consistency conditions, it is most convenient to revert to physical independent variables. We define $\underline{\mathbf{y}}^{(0)}(x, t)=\underline{\mathbf{w}}^{(0)}(\xi, \eta, \hat{t})$ and obtain

$$
\begin{align*}
& \underline{y}_{1 t}^{(0)}+\lambda_{1} \underline{y}_{1 x}^{(0)}=\epsilon^{2}\left(\frac{n_{12} n_{21}}{\lambda_{2}-\lambda_{1}} \lambda_{1}^{4}+r_{11} \lambda_{1}^{3}-s_{11} \lambda_{1}^{2}\right) \underline{y}_{1 \times x x}^{(0)}  \tag{4.13a}\\
& \underline{y}_{2 t}^{(0)}+\lambda_{2} \underline{y}_{2 x}^{(0)}=\epsilon^{2}\left(\frac{n_{12} n_{21}}{\lambda_{1}-\lambda_{2}} \lambda_{2}^{4}+r_{22} \lambda_{2}^{3}-s_{22} \lambda_{2}^{2}\right) \underline{y}_{2 \times x x}^{(0)} \tag{4.13b}
\end{align*}
$$

(To obtain (4.13a), we consider $\hat{t}$ and $\xi$ as the two independent variables that are being replaced by $x$ and $t$; for the second equation, $\hat{t}$ and $\eta$ are the two independent variables.)

Notice that these two equations are linearized Korteweg-de Vries (KdV) equations with constant coefficients. The terms with three $x$ derivatives indicate the dispersive
nature of these equations, and the $\epsilon^{2}$ premultiplying these terms shows the relative strength of the dispersive effects to the advection represented on the left-hand sides.

Given the initial conditions

$$
P \underline{\mathbf{y}}^{(0)}(x, 0)=\underline{\mathbf{u}}^{(0)}(x, 0,0)=\underline{\mathbf{h}}^{(0)}(x)=\left[\begin{array}{c}
\sin (\pi x)  \tag{4.14}\\
0
\end{array}\right]
$$

we use the dispersion relations of (4.13) to find the solution

$$
\begin{aligned}
& \underline{y}_{1}^{(0)}(x, t)=\frac{1}{2} \sin \left[\pi\left(x-\lambda_{1} t\right)-\epsilon^{2} \pi^{3}\left(\frac{n_{12} n_{21}}{\lambda_{2}-\lambda_{1}} \lambda_{1}^{4}+r_{11} \lambda_{1}^{3}-s_{11} \lambda_{1}^{2}\right) t\right] \\
& \underline{y}_{2}^{(0)}(x, t)=-\frac{1}{2} \sin \left[\pi\left(x-\lambda_{2} t\right)-\epsilon^{2} \pi^{3}\left(\frac{n_{12} n_{21}}{\lambda_{1}-\lambda_{2}} \lambda_{2}^{4}+r_{22} \lambda_{2}^{3}-s_{22} \lambda_{2}^{2}\right) t\right] .
\end{aligned}
$$

Notice that when $\epsilon$ and $t$ are small, the solution represents the initial conditions being advected at the speeds $\lambda_{1}$ and $\lambda_{2}$. Only when $t>\mathcal{O}\left(\epsilon^{-2}\right)$ do we see the dispersion having a significant effect on the wave speeds.

Using a spectral numerical technique (see Appendix B), we calculated the solution to the original equation (4.1) for these initial conditions ${ }^{1}$. To verify that equations (4.13) describe the right long-term behavior, we compare its solutions to the numerical solution. We also compare the numerical solution with the solution to (4.13) without the dispersive terms. This "non-dispersive" solution is the same solution that would be obtained if we solved (4.1) without any slow temporal or stretched spatial scales.

Figure 4.1 makes these comparisons for the linear wave equation, with

$$
\begin{aligned}
& \rho\left(x^{*}\right)=1+0.5 \cos \left(\pi x^{*}\right)-0.3 \sin \left(\pi x^{*}\right) \quad \text { and } \\
& k\left(x^{*}\right)=\frac{1}{1-0.1 \cos \left(\pi x^{*}\right)+0.25 \sin \left(\pi x^{*}\right)}
\end{aligned}
$$

and $\epsilon=0.1$. Because the period of the medium is $0.2 \pi$ and the period of the initial conditions is $2 \pi$, the solution is itself periodic with period $2 \pi$. Therefore, the

[^3]solutions are only displayed between $-1<x<1$. (This periodicity also enables us to use spectral numerical techniques, which are more accurate than comparable finite difference methods.) At $t=150$, we see the analytic solution without dispersion


Figure 4.1: Numerical solution versus two analytic solutions for the linear wave equation.
begins to deviate from the numerical solution. By $t=500$, the solution is almost completely out of phase with the numerical solution. The analytic solution to (4.13) taking into account the dispersive term looks very good even at $t=500$. Since the wave speed in this problem is approximately one, by $t=500$ almost 250 complete waves have passed through the computation domain.

We make the same comparisons for the test problem of Kevorkian and Bosley in Figure 4.2. In these simulations, we use the same initial conditions (4.14) and choose


Figure 4.2: Numerical solution versus two analytic solutions for the test problem of Kevorkian and Bosley.
$\alpha=0.5, \beta=\gamma=-0.2$, and $\delta=0.1$ with $\epsilon=0.1$. Again, notice that at $t=100$, the analytic solution that disregards dispersion begins to deviate from the true solution; by $t=600$, it cannot be used at all. (At $t=600$, the graph of $u_{2}$ without dispersion is difficult to see because it is identically zero.) The analytic solution that accounts for dispersion matches up very well at $t=100$ and at $t=600$ begins to slip off the true solution.

To summarize, we have seen that the dispersive effect of the fluctuations in $A\left(x^{*}\right)$ is real, and that the multiple-scale analysis accurately describes this behavior through long-term evolution equations (consistency conditions). If we continue our analysis to higher orders of $\epsilon$, we can determine more accurate long-term evolution equations.

### 4.1.7 Solving the homogenized equations for an initial-boundary value problem

Now we turn our attention to the initial-boundary value problem for (4.1) to demonstrate the difficulty of the missing boundary conditions. Suppose that our solution domain is the quarter space $x>0$ and $t>0$. Since one eigenvalue of $A\left(x^{*}\right)$ is always positive and the other is always negative, we can only specify one boundary condition at $x=0$ (since it is a space-like arc):

$$
\begin{equation*}
u_{1}(0, t ; \epsilon)=g(t ; \epsilon)=g^{(0)}(t)+\mathcal{O}(\epsilon) \text { for } t>0 \tag{4.15}
\end{equation*}
$$

Also, let's prescribe the initial conditions

$$
\begin{equation*}
\mathbf{u}(x, 0)=\underline{\mathbf{h}}^{(0)}(x)+\epsilon \mathbf{h}^{(1)}\left(x, x^{*}\right)+\cdots \text { for } x>0 \tag{4.16}
\end{equation*}
$$

Now we try to solve the consistency conditions (4.13) to demonstrate the difficulty of missing boundary conditions. What sorts of initial and boundary conditions are needed to solve these equations, and how many conditions do we have?

One way to figure out what conditions are needed to solve (4.13) is to use a Laplace transform in $t$ to turn these partial differential equations into a pair of thirdorder, constant coefficient, linear ordinary differential equations. We can write these two ODEs in the generic form

$$
s Y-f(x)+\lambda Y_{x}=\mu Y_{x x x},
$$

where $Y(x ; s)$ is the Laplace transform of either $\underline{y}_{1}^{(0)}$ or $\underline{y}_{2}^{(0)}$, and $s$ is the transformed variable, which must range from $\alpha-i \infty$ to $\alpha+i \infty$. (Choose $\alpha$ so that the integration contour is to the right of all singularities in the complex plane.) Since there is one derivative in time, we need one initial condition for each $\underline{y}_{i}^{(0)}$. The characteristic equation for this ordinary differential equation has three roots. If we require that the solution remains bounded as $x \rightarrow \infty$, then we need as many boundary conditions at $x=0$ as the number of roots with nonpositive real part for the given range of $s$. We
eliminate the possibility of all three roots having positive real part because this would lead to the trivial solution. The two partial differential equations (4.13) may require different numbers of boundary conditions at $x=0$, but each requires at least one.

Now let us examine what conditions are available to solve (4.13). First, the initial condition (4.16) provides us with

$$
\underline{\mathbf{y}}^{(0)}(x, 0)=P^{-1} \underline{\mathbf{u}}^{(0)}(x, 0)=P^{-1} \underline{\mathbf{h}}^{(0)}(x)
$$

for $x>0$, which is precisely what we need. Next, the boundary condition (4.15) gives

$$
\underline{u}_{1}^{(0)}(0, t)=p_{11} \underline{y}_{1}^{(0)}(0, t)+p_{12} \underline{y}_{2}^{(0)}(0, t)=g^{(0)}(t)
$$

for $t>0$. Armed with only a linear combination of the boundary conditions along $x=0$, we do not have enough information to determine a unique solution to (4.13). Notice that this difficulty does not occur for the initial-value problem on the infinite domain $-\infty<x<\infty$ because there isn't a boundary at $x=0$, and hence no boundary conditions to satisfy.

### 4.1.8 Reduced multiple-scales solution

To get around the missing boundary condition difficulty, let us revisit our assumption of which scales are present in the expansion of $\mathbf{u}(x, t ; \epsilon)$. Practically speaking, we use slow temporal or stretched spatial scales to avoid secular terms in our asymptotic expansion because secular terms limit the region of validity of our solution. Adding these slow scales gives us the freedom to eliminate potentially secular terms by enforcing consistency conditions. However, at the moment we are unable to solve these consistency conditions due to a lack of boundary information. Without slow temporal or stretched spatial scales, we cannot avoid secular terms, but at least a temporary solution can be found. From this temporary solution we can extract the "missing" boundary conditions that we need to solve the consistency conditions.

Let us illustrate this procedure for (4.1). We re-expand the state variables using the reduced set of scales, $x^{*}, x$ and $t$ :

$$
\mathbf{u}(x, t ; \epsilon)=\underline{\mathbf{r}}^{(0)}(x, t)+\epsilon \mathbf{r}^{(1)}\left(x, x^{*}, t\right)+\cdots
$$

We use $\mathbf{r}^{(i)}$ instead of $\mathbf{u}^{(i)}$ to avoid confusion with the previous multiple-scale expansion of $\mathbf{u}(x, t ; \epsilon)$. It is not necessary to repeat the work of obtaining the equations for each order of $\epsilon$ because the only change is that there are no derivatives with respect to any slow temporal scales. Notice that by writing $\underline{\mathbf{r}}^{(0)}(x, t)$ instead of $\mathbf{r}^{(0)}\left(x, x^{*}, t\right)$ in our new expansion, we have skipped the step of using the $\mathcal{O}\left(\epsilon^{-1}\right)$ system to eliminate the $x^{*}$-dependence of the leading-order solution.

Following the steps outlined in Section 4.1.2, we define the characteristic independent variables $\xi=x-\lambda_{1} t$ and $\eta=x-\lambda_{2} t$, along with the characteristic dependent variables $\underline{\mathbf{r}}^{(0)}(x, t)=P \underline{\mathbf{s}}^{(0)}(\xi, \eta)$. As in Section 4.1.6, the governing equations for $\underline{\mathbf{s}}^{(0)}$ are $\underline{s}_{1 \eta}^{(0)}=0$ and $\underline{s}_{2 \xi}^{(0)}=0$, which imply that

$$
\underline{\mathbf{s}}^{(0)}(\xi, \eta)=\left[\begin{array}{l}
\underline{s}_{1}^{(0)}(\xi) \\
\underline{s}_{2}^{(0)}(\eta)
\end{array}\right]
$$

In contrast to the analysis of the previous section, $\underline{\mathbf{s}}^{(0)}$ can be completely determined at this stage since we don't allow it to depend on $\hat{t}$. Using the $\mathcal{O}(1)$ initial conditions from (4.16),

$$
\underline{\mathbf{s}}^{(0)}(x, x)=\left[\begin{array}{l}
\underline{s}_{1}^{(0)}(x) \\
\underline{s}_{2}^{(0)}(x)
\end{array}\right]=P^{-1} \underline{\mathbf{r}}^{(0)}(x, 0)=P^{-1} \underline{\mathbf{h}}^{(0)}(x) \text { for } x>0
$$

Therefore,

$$
\begin{aligned}
& \underline{s}_{1}^{(0)}(\xi)=\frac{1}{\operatorname{det} P}\left(p_{22} \underline{h}_{1}^{(0)}(\xi)-p_{12} \underline{h}_{2}^{(0)}(\xi)\right) \text { for } \xi>0 \\
& \underline{s}_{2}^{(0)}(\eta)=-\frac{1}{\operatorname{det} P}\left(p_{21} \underline{h}_{1}^{(0)}(\eta)-p_{11} \underline{h}_{2}^{(0)}(\eta)\right) \text { for } \eta>0
\end{aligned}
$$

We don't have to worry about the determinant of $P$ being zero because $P$ is nonsingular.

Because $\eta=x-\lambda_{2} t$ and $\lambda_{2}<0$, the second state variable $\underline{s}_{2}^{(0)}(\eta)$ propagates information to the "left" in the $x-t$ plane. Furthermore, $\eta$ is always positive in the solution domain so $\underline{s}_{2}^{(0)}(\eta)$ is completely determined. On the other hand, $\xi=x-\lambda_{1} t$ and $\lambda_{1}>0$ so $\xi$ can take on positive and negative values in the solution domain. The first state variable $\underline{s}_{1}^{(0)}(\xi)$ propagates information to the "right" so the initial conditions only determine $\underline{s}_{1}^{(0)}(\xi)$ for $\xi>0$. We still have to define $\underline{s}_{1}^{(0)}(\xi)$ for $\xi<0$ using the $\mathcal{O}(1)$ boundary condition (4.15). Applying equation (4.15) gives $g^{(0)}(t)=$ $\underline{r}_{1}^{(0)}(0, t)=p_{11} \underline{s}_{1}^{(0)}\left(-\lambda_{1} t\right)+p_{12} \underline{s}_{2}^{(0)}\left(-\lambda_{2} t\right)$ for $t>0$. Therefore, $\underline{s}_{1}^{(0)}(\xi)=\frac{1}{p_{11}} g^{(0)}\left(-\xi / \lambda_{1}\right)+\frac{p_{12}}{p_{11} \operatorname{det} P}\left(p_{21} \underline{h}_{1}^{(0)}\left(\xi \lambda_{2} / \lambda_{1}\right)-p_{11} \underline{h}_{2}^{(0)}\left(\xi \lambda_{2} / \lambda_{1}\right)\right)$ for $\xi<0$. We are now able to recover the "missing" boundary condition,

$$
\underline{r}_{2}^{(0)}(0, t)=\frac{p_{21}}{p_{11}}\left(g^{(0)}(t)-\underline{h}_{1}^{(0)}\left(-\lambda_{2} t\right)\right)+\underline{h}_{2}^{(0)}\left(-\lambda_{2} t\right)
$$

If necessary, we can even calculate quantities like $\underline{r}_{2 x}^{(0)}(0, t)$ by taking a derivative of $\underline{\mathbf{r}}^{(0)}$ with respect to $x$, then setting $x=0$.

The only potential difficulty in the analysis above is if $p_{11}$ vanishes. We digress briefly to show why $p_{11}$ cannot be zero. One can show that if $p_{11}=0$, then $\alpha_{12}=0$ as well, where $\alpha_{i j}$ is the $i-j$ entry of $\left\langle A^{-1}\right\rangle^{-1}$. Furthermore, by our choice of $\lambda_{1}>0$ and $\lambda_{2}<0$, we can also infer that $\alpha_{11}=\lambda_{2}<0$. These facts imply that

$$
\underline{r}_{1 t}^{(0)}+\alpha_{11} \underline{1}_{1 x}^{(0)}=0
$$

or that the first-component of the leading-order homogenized solution advects information to the left in the $x-t$ plane with speed $\left|\lambda_{2}\right|$. That means that one cannot specify a boundary condition on the first-component, $u_{1}(0, t ; \epsilon)$. Therefore, we must assume $p_{11} \neq 0$ to avoid this nonsensical result.

The procedure outlined above for recovering boundary conditions can be repeated for the equations that arise at higher orders of $\epsilon$. Solving the $\mathcal{O}(\epsilon)$ system of equations using the reduced set of scales $x^{*}, x$, and $t$, enables us to find the missing boundary condition $r_{2}^{(1)}(0,0, t)$. This process will not be demonstrated here as the algebra grows exponentially with each order of $\epsilon$. We have developed Mathematica notebooks that are capable of performing this tedious task to any order of $\epsilon$.

For example, suppose that for the linear wave equation (4.2), with the specific functions

$$
\begin{aligned}
& \rho\left(x^{*}\right)=1+0.5 \cos \left(\pi x^{*}\right)-0.3 \sin \left(\pi x^{*}\right) \quad \text { and } \\
& k\left(x^{*}\right)=\frac{1}{1-0.1 \cos \left(\pi x^{*}\right)+0.25 \sin \left(\pi x^{*}\right)}
\end{aligned}
$$

we impose the boundary condition

$$
u_{1}(0, t ; \epsilon)=g^{(0)}(t)=1-\cos t
$$

In addition, suppose that the initial conditions are

$$
\mathbf{h}(x ; \epsilon)=\left[\begin{array}{c}
\sin \pi x \\
0
\end{array}\right]-\epsilon\left[\begin{array}{c}
\cos (\pi x)\left(0.1 \sin \left(\pi x^{*}\right)+0.25 \cos \left(\pi x^{*}\right)\right) \\
0
\end{array}\right]+\mathcal{O}\left(\epsilon^{2}\right)
$$

where the fluctuating parts are chosen so that the solution does not depend on $t^{*}=t / \epsilon$ (see Section 3.4). Using Mathematica, we calculate the missing boundary condition to be

$$
\begin{align*}
u_{2}(0, t ; \epsilon)= & 1-\cos (t)-\sin (\pi t)+\epsilon\left(\frac{\cos (\pi t)}{4}-\frac{201 \sin t}{400 \pi}\right) \\
& +\epsilon^{2}\left(\frac{52639 \pi t \cos (\pi t)}{320000}+\frac{399361 \cos t+209161 \pi^{2} \sin (\pi t)}{320000 \pi^{2}}\right)+\mathcal{O}\left(\epsilon^{3}\right) \tag{4.17}
\end{align*}
$$

Notice that the $\mathcal{O}\left(\epsilon^{2}\right)$ contribution to this recovered boundary information contains a secular term proportional to $t$. The presence of this secular term makes the expansion
valid only for $t$ roughly less than $\epsilon^{-1}$. As we will see, the region of validity is, in practice, usually larger.

Since we do not have an exact solution to the original linear wave equation, we must verify (4.17) numerically. Using CLAWPACK, we generate a numerical approximation to the solution of the original linear wave equation (4.1) with $A\left(x^{*}\right)$ defined in (4.3), and extract the values of $u_{2}(0, t)$ to compare with (4.17). (See Appendix B for more discussion about CLAWPACK.) As there is no computer large enough to represent the semi-infinite solution domain $x>0$ using finite volumes, we limit ourselves to a finite computation domain of $0<x<L$. The upper limit $L$ is chosen so that a wave entering the computation domain from the right boundary does not interfere with the part of the solution that we desire. We choose $L=110$ because we are interested in calculating $u_{2}(0, t)$ up to $t=100$ and in this problem the wave speeds are $\pm 1$. At $x=0$, we specify $u_{1}(0, t)$ using the given boundary condition, and simulate a free condition on $u_{2}(0, t)$ using zero-order extrapolation. We also use zero-order extrapolation for both $u_{1}$ and $u_{2}$ at $x=L$.

For $\epsilon=0.1$, we find that the leading order term of (4.17) matches the numerically calculated $u_{2}(0, t)$ very well for $t<10$. As $t$ gets larger, we must take into account the higher order corrections in (4.17) to get a better correspondence. Figure 4.3 compares the analytically recovered boundary information against the numeric solution for $75<t<100$. In pane (a), we compare the numeric solution with the $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ terms of (4.17) only; in pane (b), we include all terms displayed in (4.17). Notice that without the $\mathcal{O}\left(\epsilon^{2}\right)$ contribution, the analytic boundary information does not quite match the numeric solution. The $\mathcal{O}\left(\epsilon^{2}\right)$ contribution is therefore useful for $t<100$, even though it introduces a secular term.

A convincing way to verify that an expansion is asymptotically correct is to plot the absolute error of the expansion against $\epsilon$ using logarithmic scales [6]. The slope of the resulting line indicates the rate of convergence of the asymptotic solution to



Figure 4.3: Numerical verification of the recovered boundary information for an initialboundary value problem for the linear wave equation.
the numeric solution as $\epsilon \rightarrow 0$. Figure 4.4 shows the absolute error of the analytically recovered boundary information (4.17), with different terms of the asymptotic expansion included. The absolute error is measured using the discretized version of the integral

$$
\text { error }=\int_{0}^{T}\left|u_{2}^{(\mathrm{n})}(0, t)-u_{2}^{(\mathrm{a})}(0, t)\right| d t
$$

where $u^{(\mathrm{n})}$ is the numeric solution and $u^{(a)}$ is the analytic expression that we are observing. Ideally, we would compare the asymptotic expansion against the exact solution, but since an exact solution is not available, we must make do with a numerically calculated solution. Again we use CLAWPACK for our numeric solution, but this time with a spatial domain of $0<x<60$ and a spatial step size of 0.002 . We then sample
the boundary information, $u_{2}(0, t)$, in the range $0<t<50$ using 2000 points.


Figure 4.4: Rate of convergence of analytic recovered boundary information for an initial-boundary value problem for the linear wave equation.

Figure 4.4 shows a surprising result. Keeping only the leading order term in (4.17), we would normally expect the truncated error to be $\mathcal{O}(\epsilon)$ as $\epsilon \rightarrow 0$. However, we see that the absolute error of the leading order term behaves more like $\epsilon^{2}$ as $\epsilon \rightarrow 0$. This finding does not invalidate the asymptotic correctness of (4.17), it just means that the leading order term is more accurate than expected. Similarly, we see that the asymptotic expansion including the $\mathcal{O}\left(\epsilon^{2}\right)$ contribution converges to the true solution slightly faster than $\epsilon^{3}$ as $\epsilon \rightarrow 0$, which would normally expect. The asymptotic expansion including up to the $\mathcal{O}(\epsilon)$ contribution only and the expansion with contributions up to $\mathcal{O}\left(\epsilon^{3}\right)$ (not shown) converge close to their expected rates.

Now let us also recover the missing boundary conditions for a case in which the
initial conditions are zero (a signaling problem). We impose the same boundary condition

$$
u_{1}(0, t ; \epsilon)=g^{(0)}(t)=1-\cos t
$$

but instead specify that $\mathbf{u}(x, 0 ; \epsilon)=\mathbf{0}$. Under these conditions, we calculate the missing boundary condition to be

$$
\begin{equation*}
u_{2}(0, t ; \epsilon)=1-\cos t-\epsilon \frac{201 \sin t}{400 \pi}+\epsilon^{2} \frac{399361 \cos t}{320000 \pi^{2}}+\mathcal{O}\left(\epsilon^{3}\right) \tag{4.18}
\end{equation*}
$$

When we compare (4.18) to the recovered boundary information with nonzero initial conditions (4.17), we see that there are no secular terms in the recovered boundary information for the signaling problem.

Figure 4.5a shows the analytically recovered boundary information in (4.18) against its numeric counterpart, calculated solution using $\epsilon=0.2$ and a spatial step size of 0.0005 for $0<t<20$. (We used $0<x<25$ as our computational domain.) Displayed in this manner, the numeric and analytic $u_{2}(0, t)$ are indistinguishable. Figure 4.5 b shows that the $\mathcal{O}\left(\epsilon^{2}\right)$ contribution to the recovered boundary condition does indeed increase its accuracy.

Once again, we compare the recovered boundary information (4.18) against the numeric solution for various $\epsilon$ and plot the absolute error versus $\epsilon$ using a logarithmic scale. Figure 4.6 shows that the leading order term of (4.18) by itself converges to the true boundary information like $\epsilon$ as $\epsilon \rightarrow 0$, as expected. Adding the $\mathcal{O}(\epsilon)$ contribution in (4.18) accelerates this convergence to $\epsilon^{2}$ as $\epsilon \rightarrow 0$. We were not able to verify this latter result for the same range of $\epsilon$ because the numerical calculation reached its limit of accuracy for our chosen spatial step size. As a result, we were also unable to verify that including higher order terms in (4.18) improved the agreement between the analytic and numeric solutions. Reducing the spatial step size would enable us to verify these claims (because it would increase the overall accuracy of the numerical method and better resolve the fluctuations of the $A\left(x^{*}\right)$ matrix), but it would also



Figure 4.5: Numerical verification of the recovered boundary information for a signaling problem for the linear wave equation.
increase computation time.
Now that we have seen how to use a reduced set of multiple scales to recover boundary information that is asymptotically correct, we turn to the most striking feature of (4.17): the presence of secular terms proportional to $t$. What is the mechanism that causes these secular terms to appear in the recovered boundary information? By neglecting slow temporal and stretched spatial scales, secular terms (proportional to $x$ and $t$ ) will arise in the solution. It is therefore not surprising that secular terms proportional to $t$ may remain after we substitute $x=0$.

What is more interesting is why the secular terms go away when the initial con-


Figure 4.6: Rate of convergence of analytic recovered boundary information for a signaling problem for the linear wave equation.
ditions are zero. Using Mathematica, we have calculated the recovered boundary information to high orders of $\epsilon$. For the linear wave equation (4.2), there is only one way for the boundary information to be free of secular terms. For convenience, let us normalize $\rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$ so that $\langle\rho\rangle=\langle 1 / k\rangle=1$, so the wave speeds are $\lambda_{1,2}= \pm 1$. In this case, only if $\underline{h}_{1}^{(0)}(x)=\underline{h}_{2}^{(0)}(x)$ for all $x>0$ will the recovered boundary information be free of secular terms.

If we return to the reduced multiple-scale analysis introduced in this section, we see that the leading order behavior of the solution is $\underline{\mathbf{r}}^{(0)}(x, t)=P^{(0)}(\xi, \eta)$, where $\underline{s}_{1}^{(0)}=\underline{s}_{1}^{(0)}(\xi)$ and $\underline{s}_{2}^{(0)}=\underline{s}_{2}^{(0)}(\eta)$ are the characteristic dependent variables, and $\xi=$ $x-t$ and $\eta=x+t$ are the characteristic independent variables. Because of our choice of variables, $\underline{s}_{1}^{(0)}$ represents a wave traveling to the "right" in the $x-t$ plane,
and $\underline{s}_{2}^{(0)}$ represents a wave traveling to the "left." With $\rho$ and $k$ normalized, $\underline{s}_{2}^{(0)}(\eta)=$ $\underline{h}_{2}^{(0)}(\eta)-\underline{h}_{1}^{(0)}(\eta)$ for all $\eta>0$, so if $\underline{h}_{1}^{(0)}(x)=\underline{h}_{2}^{(0)}(x)$ then $\underline{s}_{2}^{(0)}=0$. Therefore, we see that the recovered boundary information is free of secular terms only when the wave traveling to the left in the $x$ - $t$ plane is identically zero.

Why would the recovered boundary information be free of secular terms when the initial conditions are such that the left-going wave vanishes? The reason is that without the slow and stretched scales, the multiple-scale analysis produces a solution that models advection only. This purely advected solution is only accurate for small $t$ because the true long-term behavior of the solution includes dispersion or higherorder effects. The recovered boundary information is the result of the interaction between the partial differential equation, the given boundary condition, and the leftgoing wave hitting the boundary. If the left-going wave is nonzero, the recovered boundary information will become inaccurate for large $t$. Therefore, the presence of secular terms is merely a sign that the recovered boundary information cannot be trusted for large $t$. Neglecting these secular terms not only makes the recovered boundary information less accurate (as in the example initial-boundary value problem in this section), it also gives the false impression that the recovered boundary information can be used for large $t$. However, the reader should keep in mind that for the signaling problem (initial conditions equal to zero), the left-going wave is zero, and the recovered boundary information will have no secular terms. We take advantage of this fact in the next section.

### 4.1.9 Multiple scales solution revisited

Now armed with the recovered boundary information, we can return to solving the consistency conditions (4.13). A general solution to these equations would be very complicated, so we demonstrate the solution procedure for the linear wave equation
with

$$
\begin{aligned}
& \rho\left(x^{*}\right)=1+0.5 \cos \left(\pi x^{*}\right)-0.3 \sin \left(\pi x^{*}\right) \quad \text { and } \\
& k\left(x^{*}\right)=\frac{1}{1-0.1 \cos \left(\pi x^{*}\right)+0.25 \sin \left(\pi x^{*}\right)}
\end{aligned}
$$

the initial conditions

$$
\mathbf{u}(x, 0 ; \epsilon)=P^{-1} \underline{\mathbf{h}}^{(0)}+\mathcal{O}(\epsilon)=\left[\begin{array}{c}
\frac{1}{2} \sin (\pi x) \\
-\frac{1}{2} \sin (\pi x)
\end{array}\right]+\mathcal{O}(\epsilon)
$$

and the boundary information

$$
u_{1}(0, t ; \epsilon)=g(t ; \epsilon)=1-\cos t+\mathcal{O}(\epsilon) .
$$

Since the original partial differential equation (4.1) is linear, we can use superposition to partition the initial-boundary value problem into two problems, an initial-value problem, and a signaling problem. Let $\mathbf{u}(x, t ; \epsilon)=\mathbf{u}^{(A)}(x, t ; \epsilon)+\mathbf{u}^{(B)}(x, t ; \epsilon)$, where $\mathbf{u}^{(A)}$ and $\mathbf{u}^{(B)}$ satisfy the same partial differential equation (4.1), but $\mathbf{u}^{(A)}$ satisfies the initial-value problem in the infinite domain and $\mathbf{u}^{(B)}$ satisfies the signaling problem in the semi-infinite domain. Specifically, $\mathbf{u}^{(A)}(x, 0 ; \epsilon)=\mathbf{h}(x ; \epsilon)$ on $-\infty<x<\infty$, while $u_{1}^{(B)}(0, t ; \epsilon)=g(t ; \epsilon)-u_{1}^{(A)}(0, t ; \epsilon)$ for $t>0$ and $\mathbf{u}^{(B)}(x, 0 ; \epsilon)=0$ for $x>0$. It doesn't matter how the initial conditions $\mathbf{h}(x ; \epsilon)$ are extended for negative $x$, because once the solution to the initial-value problem is obtained, $u_{1}^{(A)}(0, t ; \epsilon)$ is subtracted from the boundary information, $u_{1}^{(B)}(0, t ; \epsilon)$. The sum of $\mathbf{u}^{(A)}$ and $\mathbf{u}^{(B)}$ will satisfy all of the original initial and boundary conditions.

There are two reasons why we want to consider a signaling problem instead of a problem with nontrivial initial and boundary conditions. The first reason is that we have seen that the recovered boundary conditions contain secular terms when the initial conditions are nontrivial. To maximize the region of validity of our asymptotic expansion, we should use the recovered boundary conditions for the signaling problem
instead. The second reason is that solving the long-term evolution equations is much simpler with zero initial conditions.

The initial-value problem for $\mathbf{u}^{(A)}$ has already been solved in Section 4.1.6. If we extend the initial conditions in the natural way for negative $x$, the solution for our particular choice of $\epsilon=0.1, \rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$ is

$$
\mathbf{u}^{(A)}(x, t ; \epsilon)=\frac{1}{2}\left[\begin{array}{c}
\sin \left(\frac{31947361 \pi t}{32000000}+\pi x\right)-\sin \left(\frac{31947361 \pi t}{32000000}-\pi x\right) \\
-\sin \left(\frac{31947361 \pi t}{32000000}+\pi x\right)-\sin \left(\frac{31947361 \pi t}{32000000}-\pi x\right)
\end{array}\right]+\mathcal{O}(\epsilon)
$$

Now we are ready to solve the signaling problem for $\mathbf{u}^{(B)}$. Let $\mathbf{u}^{(B)}(x, t ; \epsilon)=$ $P \underline{\mathbf{y}}^{(B)}(x, t)+\mathcal{O}(\epsilon)$, and $\underline{\mathbf{y}}^{(B)}$ satisfy the long-term evolution equations (4.13). The boundary conditions for $\underline{\mathbf{y}}^{(B)}(0, t)$ come from the recovered boundary conditions for the signaling problem (4.18) and $\mathbf{u}^{(A)}(0, t ; \epsilon)$ :

$$
\begin{aligned}
\underline{\mathbf{y}}^{(B)}(0, t) & =P^{-1} \mathbf{u}^{(B)}(0, t ; 0)=P^{-1}\left[\begin{array}{c}
g^{(0)}(t)-u_{1}^{A}(0, t ; 0) \\
1-\left(\frac{399361}{32000000 \pi^{2}}-1\right) \cos t-\frac{201}{4000 \pi} \sin t
\end{array}\right] \\
& =\left[\begin{array}{c}
1+\left(\frac{399361}{64000000 \pi^{2}}-1\right) \cos t-\frac{201}{8000 \pi} \sin t \\
\frac{39961 \cos t-1608000 \pi \sin t}{64000000 \pi^{2}}
\end{array}\right]
\end{aligned}
$$

To solve for $\underline{\mathbf{y}}^{(B)}$, we use a Laplace transform in $t$ to turn the partial differential equations (4.13) into third-order ordinary differential equations:

$$
\begin{align*}
&-\frac{52639}{32000000 \pi^{2}} \frac{d^{3} \underline{Y}_{1}^{(B)}}{d x^{3}}-\frac{d \underline{Y}_{1}^{(B)}}{d x}-s \underline{Y}_{1}^{(B)}=0  \tag{4.19a}\\
& \frac{52639}{32000000 \pi^{2}} \frac{d^{3} \underline{Y}_{2}^{(B)}}{d x^{3}}+\frac{d \underline{Y}_{2}^{(B)}}{d x}-s \underline{Y}_{1}^{(B)}=0 \tag{4.19b}
\end{align*}
$$

where

$$
\underline{Y}_{i}^{(B)}(x, s)=\mathcal{L}\left[\underline{y}_{i}^{(B)}(x, t)\right]=\int_{0}^{\infty} e^{-s t} \underline{y}_{i}^{(B)}(x, t) d t
$$

Using Mathematica, we have verified that only one of the three roots of the characteristic equation for $\underline{Y}_{1}^{(B)}$ has nonpositive real part, and two of the roots of the
characteristic equation for $\underline{Y}_{2}^{(B)}$ have nonpositive real part. Let these roots be $\sigma_{1}^{(1)}$ and $\sigma_{1}^{(2)}, \sigma_{2}^{(2)}$, respectively; we don't display these roots here for brevity. We do not use the homogeneous solutions based on the roots with positive real parts because we want the solution to remain bounded as $x \rightarrow \infty$. The fact that there are two roots with nonpositive real part for $\underline{Y}_{2}^{(B)}$ means that we need two boundary conditions to find a unique solution for $\underline{Y}_{2}^{(B)}$. In the previous section, we explained how to obtain the extra information that we need, $\underline{y}_{2 x}^{(B)}(0, t)$. We don't display this recovered boundary information to save space.

Once we have recovered all the necessary boundary conditions, the solutions to the ordinary differential equations (4.19a) and (4.19b) are

$$
\begin{aligned}
\underline{Y}_{1}^{(B)}(x, t)= & \mathcal{L}\left[\underline{y}_{1}^{(B)}(0, t)\right] \exp \left(\sigma_{1}^{(1)} x\right) \\
\underline{Y}_{2}^{(B)}(x, t)= & \mathcal{L}\left[\underline{y}_{2}^{(B)}(0, t)\right] \frac{\sigma_{1}^{(2)} \exp \left(\sigma_{2}^{(2)} x\right)-\sigma_{2}^{(2)} \exp \left(\sigma_{1}^{(2)} x\right)}{\sigma_{1}^{(2)}-\sigma_{2}^{(2)}} \\
& +\mathcal{L}\left[\underline{y}_{2 x}^{(B)}(0, t)\right] \frac{\exp \left(\sigma_{1}^{(2)} x\right)-\exp \left(\sigma_{2}^{(2)} x\right)}{\sigma_{1}^{(2)}-\sigma_{2}^{(2)}} .
\end{aligned}
$$

The difficult part is now the inversion of these expressions. As the roots $\sigma_{1}^{(1)}, \sigma_{1}^{(2)}$, and $\sigma_{2}^{(2)}$ involve cube and square roots of $s$, the calculation of the Laplace inverse involves branch cuts. As a result, the inversion integrals cannot be expressed in closed-form and have to be approximated either analytically or numerically.

One way to make progress with an analytic approximation is to use residue calculus and Watson's Lemma (see Section 2.1 and Chapter 4 of [26]). In particular, we use the result that if $F(s)$ has a branch point at $s_{0}$ and can be expressed there as

$$
F(s)=\left(s-s_{0}\right)^{\gamma} \sum_{n=0}^{\infty} a_{n}\left(s-s_{0}\right)^{n}
$$

with $a_{0} \neq 0$ and $\gamma>-1$, then

$$
\mathcal{L}^{-1}[F(s)]=\frac{1}{2 \pi i} \int_{\alpha-i \infty}^{\alpha+i \infty} F(s) e^{t s} d s \sim-\frac{e^{t s_{0}} \sin (\gamma \pi)}{\pi t^{\gamma+1}} \sum_{n=0}^{\infty} a_{n}(-1)^{n} t^{-n} \Gamma(\gamma+n+1)
$$

as $t \rightarrow \infty$. In many cases, the dominant term of this asymptotic expansion is sufficient for large $t$.

We locate and expand about the branch points of $\underline{Y}_{1}^{(B)}$ and $\underline{Y}_{2}^{(B)}$. Residue calculus and the dominant term of the branch point expansion give

$$
\begin{align*}
& \underline{y}_{1}^{(B)}(x, t) \approx 1-0.999368 \cos (t-1.00017 x)-.00799754 \sin (1-1.00017 x) \\
&+x t^{-3 / 2}[ 6.81765 \times 10^{-5} \cos (29.8139 t-44.7209 x)  \tag{4.20}\\
&\left.-2.01145 \times 10^{-5} \sin (29.8193 t-44.7209 x)\right]
\end{align*}
$$

Notice that the first two terms of (4.20) are the most significant terms, and that they model the advection of the boundary information, $\underline{y}_{1}^{(B)}(0, t)$, with a small amount of dispersion. We don't show $\underline{y}_{2}^{(B)}$ because it is too long. The coefficients in the expansions of $\underline{Y}_{1}^{(B)}$ and $\underline{Y}_{2}^{(B)}$ about its branch points involve increasing powers of $x$ since we are essentially seeking expansions of $\exp (\sigma x)$. If $x$ is large and the real part of $\sigma$ is negative, the power series expansion of $\exp (\sigma x)$ requires many terms before the series begins to converge. This means that the approximation given above is only valid for large $t$ and small $x$.

### 4.1.10 Summary and discussion

We have seen that for the linear wave equation and the test problem by Kevorkian and Bosley, the cumulative (long-term) effects of the $x^{*}$-scale fluctuations in $A\left(x^{*}\right)$ only appear on the $\hat{t}=\epsilon^{2} t$ scale. This fact is a direct consequence of the diagonal entries of $\mathcal{N}$ being zero and it implies that the long-term evolution of the solution does not include diffusion.

We believe that it is more than coincidence that diffusion should be absent for solutions to remain bounded. The reader should keep in mind that the dependent variables u come from a perturbation to a steady state solution for an arbitrary conservation law. So if $\mathbf{u}$ grows in time, its corresponding steady state solution is unstable. We
believe that for all physically realistic situations in which the steady state solution is stable (implying that $\mathbf{u}$ remains bounded), the homogenized solution should not diffuse on the $\tilde{t}$ or $\tilde{x}$ scales.

As we will see in the next section, the presence of diffusion depends solely on the matrix $A\left(x^{*}\right)$. With some algebra, one can compute that diffusion will be absent if

$$
\begin{equation*}
\left\langle b_{12} \llbracket b_{21} \rrbracket\right\rangle\left\langle b_{11}-b_{22}\right\rangle+\left\langle\left(b_{11}-b_{22}\right) \llbracket b_{12} \rrbracket\right\rangle\left\langle b_{21}\right\rangle+\left\langle b_{21} \llbracket b_{11}-b_{22} \rrbracket\right\rangle\left\langle b_{12}\right\rangle=0, \tag{4.21}
\end{equation*}
$$

where $b_{i j}\left(x^{*}\right)$ is the $i-j$ entry of $A^{-1}\left(x^{*}\right)$. Notice that this condition is easily satisfied if either $b_{11}=b_{22}$ or $b_{12}=b_{21}$. (The latter fact and the computation of this condition requires the fact that $\langle a \llbracket b \rrbracket\rangle=-\langle b \llbracket a \rrbracket\rangle$, which is proved in Appendix A.) In all of the physical examples we have seen, one of these simpler conditions is satisfied. For the test problem of Kevorkian and Bosley (a nonphysical problem), condition (4.21) is equivalent to the condition $(\alpha-\delta)(\beta-\gamma)=0$ that we computed in Section 3.3.

Assuming that diffusion is absent from the consistency conditions, the next observable long-term phenomenon is dispersion. Problems that exhibit dispersion and not diffusion have a pair of decoupled, linear KdV equations as their long-term evolution equations (consistency conditions). For the solution domain $x>0$ and $t>0$, these PDEs require one or two boundary conditions along $x=0$ and one initial condition along $t=0$ for each dependent variable. Because our original hyperbolic problem only allows us to specify a linear combination of both state variables along the boundary $x=0$, we cannot solve the long-term evolution equations armed with only the information necessary for a well-posed configuration of the original problem (4.1).

In Section 4.1.8, we showed that this problem can be circumvented by temporarily ignoring the dependence of the solution on any slow temporal scales during the multiple-scale analysis. Without these scales, we avoid those problematic consistency conditions at the expense of a solution that is asymptotically valid in a smaller region of the solution domain. Once this temporary solution has been obtained, we can
extract the "missing" boundary conditions and return to the original multiple-scale homogenization technique.

We have also seen that the recovered boundary information will contain secular terms proportional to $t$ when the initial conditions are nontrivial. These presence of these secular terms restricts the region of validity of the calculated solution. Fortunately, the signaling problem is one situation in which the recovered boundary information does not contain secular terms. In Section 4.1.9, we employed this fact to our advantage by using superposition to partition the problem into two pieces, an initialvalue problem on the semi-infinite domain, and a signaling problem with zero initial conditions. The initial-value problem is solved using multiple scales, including slow temporal scales. The consistency conditions for the signaling problem are solved using the recovered boundary information from a multiple-scale analysis with a reduced set of scales. In this manner, we are able to construct a solution that is asymptotically valid for all time.

### 4.2 The general case

We now turn our attention to the general nonlinear problem represented by

$$
\begin{equation*}
\mathbf{u}_{t}+A\left(x^{*}\right) \mathbf{u}_{x}+B\left(x^{*}\right) \mathbf{u}=\epsilon\left[C\left(\mathbf{u}, x^{*}\right) \mathbf{u}+D\left(\mathbf{u}, x^{*}\right) \mathbf{u}_{x}\right]+\mathcal{O}\left(\epsilon^{2}\right) \tag{4.22}
\end{equation*}
$$

which we introduced in Section 3.1. Recall that (4.22) arises when we linearize a general system of conservation laws about its steady state solution. In this section, we discuss the solution of (4.22) when $\mathbf{u}$ is a vector of two unknown functions and when both initial and boundary conditions are present.

Suppose the initial conditions for (4.22) are

$$
\begin{equation*}
\mathbf{u}(x, 0 ; \epsilon)=\underline{\mathbf{h}}^{(0)}(x)+\epsilon \mathbf{h}^{(1)}\left(x, x^{*}\right)+\mathcal{O}\left(\epsilon^{2}\right) \tag{4.23}
\end{equation*}
$$

where the fluctuating parts of higher order terms cannot be specified arbitrarily if we
want to ensure a solution that is independent of the fast time variable, $t^{*}=t / \epsilon$ (see Section 3.4).

Equation (4.22) is a system of first-order partial differential equations and its leading order behavior is governed by the linear terms on the left-hand side. Following the discussion in Section 3.2, we require $A\left(x^{*}\right)$ to be invertible and further assume that one eigenvalue is always positive and the other always negative. This assumption allows us to specify only one boundary condition at $x=0$. The most general way of writing this boundary condition is to specify a linear combination of the state variables,

$$
\begin{equation*}
\alpha u_{1}(0, t ; \epsilon)+\beta u_{2}(0, t ; \epsilon)=g(t ; \epsilon)=g^{(0)}(t)+\mathcal{O}(\epsilon) \tag{4.24}
\end{equation*}
$$

for $t>0$. However, we do not lose any generality by assuming that $\alpha=1$ and $\beta=0$. In other words, we may assume that the first dependent variable is specified at the boundary $x=0$ and the second dependent variable is left unspecified. The reason we can do this is that for any constant $2 \times 2$ nonsingular matrix $Z$, the substitution $\mathbf{u}=Z \mathbf{v}$ does not fundamentally change the form of our problem. This substitution changes (4.22) into

$$
\begin{aligned}
\mathbf{v}_{t}+Z^{-1} A\left(x^{*}\right) Z \mathbf{v}_{x}+Z^{-1} B\left(x^{*}\right) Z \mathbf{v}= & \epsilon\left[Z^{-1} C\left(Z \mathbf{v}, x^{*}\right) Z \mathbf{v}+Z^{-1} D\left(Z \mathbf{v}, x^{*}\right) Z \mathbf{v}_{x}\right] \\
& +\mathcal{O}\left(\epsilon^{2}\right)
\end{aligned}
$$

which is still of the form represented by (4.22). However, the substitution changes the boundary condition (4.24) to

$$
\left[\begin{array}{ll}
\alpha & \beta
\end{array}\right] Z \mathbf{v}(0, t)=g^{(0)}(t)+\epsilon g^{(1)}(t)+\cdots
$$

so we can use it to alter the linear combination of $u_{1}(0, t ; \epsilon)$ and $u_{2}(0, t ; \epsilon)$.
Suppose the solution to (4.22) has the asymptotic expansion

$$
\mathbf{u}(x, t ; \epsilon)=\mathbf{u}^{(0)}\left(x, x^{*}, \tilde{x}, t, \tilde{t}\right)+\epsilon \mathbf{u}^{(1)}\left(x, x^{*}, \tilde{x}, t, \tilde{t}\right)+\mathcal{O}\left(\epsilon^{2}\right)
$$

We include the slow and stretched scales $\tilde{x}=\epsilon x$ and $\tilde{t}=\epsilon t$ because we only intend carrying out the analysis to the $\mathcal{O}(\epsilon)$ system of equations-if we want to continue the analysis to $\mathcal{O}\left(\epsilon^{2}\right)$, we must first calculate the $\mathcal{O}\left(\epsilon^{2}\right)$ contribution in (4.22). However, the solution method does not change significantly by including more spatial or temporal scales.

As before, the multiple-scale analysis proceeds by plugging into (4.22) the expressions

$$
\begin{aligned}
& \mathbf{u}_{t} \rightarrow \mathbf{u}_{t}^{(0)}+\epsilon\left(\mathbf{u}_{t}^{(1)}+\mathbf{u}_{\tilde{t}}^{(0)}\right)+\cdots \quad \text { and } \\
& \mathbf{u}_{x} \rightarrow \epsilon^{-1} \mathbf{u}_{x^{*}}^{(0)}+\left(\mathbf{u}_{x}^{(0)}+\mathbf{u}_{x^{*}}^{(1)}\right)+\epsilon\left(\mathbf{u}_{\tilde{x}}^{(0)}+\mathbf{u}_{x}^{(1)}+\mathbf{u}_{x^{*}}^{(2)}\right)+\cdots
\end{aligned}
$$

and collecting terms according to their powers of $\epsilon$.

### 4.2.1 $\mathcal{O}\left(\epsilon^{-1}\right)$ system

When we collect all terms that are proportional to $\epsilon^{-1}$, we find that

$$
A\left(x^{*}\right) \mathbf{u}_{x^{*}}^{(0)}=\mathbf{0}
$$

Since the eigenvalues of $A\left(x^{*}\right)$ are never zero, $A\left(x^{*}\right)$ may be inverted to obtain $\mathbf{u}_{x^{*}}^{(0)}=$ 0. In other words, the leading-order behavior of the solution does not depend on the fast spatial scale: $\mathbf{u}^{(0)}=\underline{\mathbf{u}}^{(0)}(x, \tilde{x}, t)$. (As before, we use the convention that all quantities independent of the fast spatial scale are underlined.)

### 4.2.2 $\mathcal{O}(1)$ system

The $\mathcal{O}(1)$ system is

$$
\begin{equation*}
\mathbf{u}_{x^{*}}^{(1)}=-A^{-1} \underline{\mathbf{u}}_{t}^{(0)}-A^{-1} B \underline{\mathbf{u}}^{(0)}-\underline{\mathbf{u}}_{x}^{(0)} . \tag{4.25}
\end{equation*}
$$

Before we integrate, we remove terms that are independent of $x^{*}$ using the averaging operators (see Appendix A). We obtain the $x^{*}$-homogenized equation

$$
\begin{equation*}
\underline{\mathbf{u}}_{t}^{(0)}+\left\langle A^{-1}\right\rangle^{-1} \underline{\mathbf{u}}_{x}^{(0)}+\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B\right\rangle \underline{\mathbf{u}}^{(0)}=\mathbf{0} . \tag{4.26}
\end{equation*}
$$

When we integrate the remaining part of (4.25) with respect to $x^{*}$, we obtain

$$
\begin{equation*}
\mathbf{u}^{(1)}=-\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t}^{(0)}-\llbracket A^{-1} B \rrbracket \underline{\mathbf{u}}^{(0)}+\underline{\mathbf{u}}^{(1)}(x, \tilde{x}, t, \tilde{t}) \tag{4.27}
\end{equation*}
$$

where $\underline{\mathbf{u}}^{(1)}$ is the constant of integration.
Equation (4.27) also tells us that for a $t^{*}$-independent solution, we must define $\mathbf{h}^{(1)}\left(x, x^{*}\right)=\llbracket A^{-1} \rrbracket\left\langle A^{-1}\right\rangle^{-1}\left(\frac{d \underline{\mathbf{h}}^{(0)}(x)}{d x}+\left\langle A^{-1} B\right\rangle \underline{\mathbf{h}}^{(0)}(x)\right)-\llbracket A^{-1} B \rrbracket \underline{\mathbf{h}}^{(0)}+\underline{\mathbf{h}}^{(1)}(x)$, where $\underline{\mathbf{h}}^{(1)}(x)$ can be specified arbitrarily, but the fluctuating part of $\mathbf{h}^{(1)}\left(x, x^{*}\right)$ cannot.

### 4.2.3 $\mathcal{O}(\epsilon)$ system

The $\mathcal{O}(\epsilon)$ system is

$$
\begin{align*}
\mathbf{u}_{x^{*}}^{(2)}= & -\mathbf{u}_{x}^{(1)}-\underline{\mathbf{u}}_{\tilde{x}}^{(0)}-A^{-1} \mathbf{u}_{t}^{(1)}-A^{-1} \underline{\mathbf{u}}_{\tilde{t}}^{(0)}-A^{-1} B \mathbf{u}^{(1)} \\
& +A^{-1} C\left(\underline{\mathbf{u}}^{(0)}, x^{*}\right) \underline{\mathbf{u}}^{(0)}+A^{-1} D\left(\underline{\mathbf{u}}^{(0)}, x^{*}\right) \underline{\mathbf{u}}_{x}^{(0)} . \tag{4.28}
\end{align*}
$$

We plug in (4.27) and remove $x^{*}$-independent terms to obtain

$$
\begin{align*}
\underline{\mathbf{u}}_{t}^{(1)}+\left\langle A^{-1}\right\rangle^{-1} & \underline{\mathbf{u}}_{x}^{(1)}+\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B\right\rangle \underline{\mathbf{u}}^{(1)}=-\left\langle A^{-1}\right\rangle^{-1} \underline{\mathbf{u}}_{\tilde{x}}^{(0)}-\underline{\mathbf{u}}_{\tilde{t}}^{(0)} \\
& +\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \rrbracket\right\rangle \underline{\mathbf{u}}_{t t}^{(0)}+\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} B \rrbracket\right\rangle \underline{\mathbf{u}}_{t}^{(0)} \\
& +\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B \llbracket A^{-1} \rrbracket\right\rangle \underline{\mathbf{u}}_{t}^{(0)}+\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B \llbracket A^{-1} B \rrbracket\right\rangle \underline{\mathbf{u}}^{(0)} \\
& +\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} C\left(\underline{\mathbf{u}}^{(0)}, x^{*}\right)\right\rangle \underline{\mathbf{u}}^{(0)}+\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} D\left(\underline{\mathbf{u}}^{(0)}, x^{*}\right)\right\rangle \underline{\mathbf{u}}_{x}^{(0)} . \tag{4.29}
\end{align*}
$$

When we integrate the remaining part of (4.28) with respect to $x^{*}$, we obtain

$$
\begin{align*}
\mathbf{u}^{(2)}= & \llbracket \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t x}^{(0)}+\llbracket \llbracket A^{-1} B \rrbracket \rrbracket \underline{\mathbf{u}}_{x}^{(0)}+\llbracket A^{-1} \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t t}^{(0)}+\llbracket A^{-1} \llbracket A^{-1} B \rrbracket \rrbracket \underline{\mathbf{u}}_{t}^{(0)} \\
& +\llbracket A^{-1} B \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{u}}_{t}^{(0)}+\llbracket A^{-1} B \llbracket A^{-1} B \rrbracket \rrbracket \underline{\mathbf{u}}^{(0)}+\llbracket A^{-1} C\left(\underline{\mathbf{u}}^{(0)}, x^{*}\right) \rrbracket \underline{\mathbf{u}}^{(0)} \\
& +\llbracket A^{-1} D\left(\underline{\mathbf{u}}^{(0)}, x^{*}\right) \rrbracket \underline{\mathbf{u}}_{x}^{(0)}-\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t}^{(1)}-\llbracket A^{-1} B \rrbracket \underline{\mathbf{u}}^{(1)}+\underline{\mathbf{u}}^{(2)}(x, \tilde{x}, t, \tilde{t}), \tag{4.30}
\end{align*}
$$

where $\underline{\mathbf{u}}^{(2)}$ is the constant of integration. By substituting $x=0$ into (4.30), we can obtain the specific form of the $\mathcal{O}\left(\epsilon^{2}\right)$ initial conditions that allows for a $t^{*}$-independent solution. We don't display it for brevity.

### 4.2.4 Solving the homogenized equations

We now turn our attention to the homogenized systems of equations obtained from the previous analysis. To solve (4.26), we diagonalize $\left\langle A^{-1}\right\rangle^{-1}$ by defining $\left\langle A^{-1}\right\rangle^{-1}=$ $P \wedge P^{-1}$, where $\Lambda=\operatorname{diag}\left[\lambda_{1}, \lambda_{2}\right]$. We choose $\lambda_{1}>0$ and $\lambda_{2}<0$ without any loss of generality and define the characteristic independent variables $\xi=x-\lambda_{1} t$ and $\eta=x-\lambda_{2} t$ along with the characteristic dependent variables $\underline{\mathbf{w}}^{(i)}=P^{-1} \underline{\mathbf{u}}^{(i)}$. We change derivatives using the formulas $\partial_{t}=-\lambda_{1} \partial_{\xi}-\lambda_{2} \partial_{\eta}$ and $\partial_{x}=\partial_{\xi}+\partial_{\eta}$. All of these substitutions result in

$$
\mathcal{L}^{(\mathrm{h})}\left[\underline{\mathbf{w}}^{(0)}\right] \stackrel{\text { def }}{=}\left[\begin{array}{l}
\left(\lambda_{1}-\lambda_{2}\right) \underline{w}_{1 \eta}^{(0)}  \tag{4.31}\\
\left(\lambda_{2}-\lambda_{1}\right) \underline{w}_{2 \xi}^{(0)}
\end{array}\right]+P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B\right\rangle \underline{\mathbf{w}}^{(0)}=\mathbf{0} .
$$

The differential operator $\mathcal{L}^{(h)}$ governs the $x^{*}$-homogenized equations at every order of $\epsilon$. We don't find out how $\underline{\mathbf{u}}^{(0)}$ depends on the slower scales $\tilde{x}$ or $\tilde{t}$ until we consider the set of equations at the next order of $\epsilon$.

Now the reason why we examined $B\left(x^{*}\right)=0$ case in Section 4.1 becomes apparentunless one of the off-diagonal terms of $P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B\right\rangle P$ vanishes, the equations in (4.31) are essentially coupled and their solution can only be written in terms of integrals of Bessel functions (see Section 3.7.1 of [18]).

The $\mathcal{O}(\epsilon)$ system of $x^{*}$-homogenized equations (4.29), written using the charac-
teristic dependent variables, is

$$
\begin{align*}
\mathcal{L}^{(\mathrm{h})}\left[\underline{\mathbf{w}}^{(1)}\right]=-\Lambda \underline{\mathbf{w}}_{\tilde{x}}^{(0)}-\underline{\mathbf{w}}_{\tilde{t}}^{(0)} & +P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} \rrbracket\right\rangle P \underline{\mathbf{w}}_{t t}^{(0)} \\
& +P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} \llbracket A^{-1} B \rrbracket\right\rangle P \underline{\mathbf{w}}_{t}^{(0)} \\
& +P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B \llbracket A^{-1} \rrbracket\right\rangle P \underline{\mathbf{w}}_{t}^{(0)} \\
& +P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B \llbracket A^{-1} B \rrbracket\right\rangle P \underline{\mathbf{w}}^{(0)}  \tag{4.32}\\
& +P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} C\left(P \underline{\mathbf{w}}^{(0)}, x^{*}\right)\right\rangle P_{\mathbf{w}^{(0)}} \\
& +P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} D\left(P \underline{\mathbf{w}}^{(0)}, x^{*}\right)\right\rangle P \underline{\mathbf{w}}_{x}^{(0)} .
\end{align*}
$$

Notice that (4.32) is essentially the same as (4.31), except with source terms. At this stage, one has to remove terms that could potentially lead to secular terms when solving for $\underline{\mathbf{w}}^{(1)}$. In the linear problem of Section 4.1, this task was easy to do because solving the $x^{*}$-homogenized operator only involved integrating the first component with respect to $\eta$ and the second with respect to $\xi$. For the general case, we must remove all terms from the right hand side of (4.32) that are homogeneous solutions of the homogenized operator $\mathcal{L}^{(h)}$. It is easy to see that $\underline{\mathbf{w}}^{(0)}, \underline{\mathbf{w}}_{\tilde{x}}^{(0)}$ and $\underline{\mathbf{w}}_{\tilde{t}}^{(0)}$ are in the null space of $\mathcal{L}^{(h)}$, but it is difficult to see which of the remaining terms need to be removed.

Let's look at the different possibilities of consistency conditions that can arise. In the most degenerate case when all the matrices on the right hand side of (4.32) are zero, the consistency conditions will indicate that $\tilde{x}$ and $\tilde{t}$ should be replaced with $\hat{x}=\epsilon^{2} x$ and $\hat{t}=\epsilon^{2}$, and that the analysis must be carried out to $\mathcal{O}\left(\epsilon^{2}\right)$ to see any long-term effects.

The other extreme occurs when every term on the right hand side of (4.32) contributes to the consistency condition. In this scenario, the consistency conditions will be a pair of coupled, constant-coefficient, Burgers' equations. The presence of two derivatives with respect to $t$ indicates that diffusion is involved, and that we will not have enough boundary conditions to solve the consistency conditions. (When the con-
sistency conditions are written in physical variables, the terms with two $t$ derivatives become spatial or temporal derivatives depending on whether $\tilde{x}$ or is involved.) We would have to recover the boundary information using a reduced set of multiple scales (see Section 4.1.8 or Section 5.1.7). However, notice that in (4.32) the matrix $A\left(x^{*}\right)$ is solely responsible for the presence of terms with two $t$ derivatives. In Section 4.1.10 we determined the condition for the diagonal entries of $P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1}\left[A^{-1}\right]\right\rangle P$ vanish. We do not know of any physical problems in which this condition is violated.

This observation leads us to believe that the most likely possibility is the one between these two extremes, in which the consistency conditions become a pair of coupled, constant-coefficient, quasi-linear, first-order partial differential equations. In this situation, the fact that the consistency condition and the original PDE have the order of derivatives in $x$ and $t$ suggests that missing boundary conditions is not a difficulty. But lest we think we can have our banana flambé and eat it too, the absence of terms with two derivatives in $t$ means that there is no diffusion to regulate the onset of shocks. As we discussed in Section 3.4, the moment a shock forms we must include $t^{*}=t / \epsilon$ in our asymptotic expansion and a general solution will be out of reach.

Let us illustrate these ideas with an example. The governing equations for elastic waves in a one-dimensional solid are

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\rho\left(x^{*}\right) V\right)-\frac{\partial}{\partial x}(S(F)) & =0 \\
\frac{\partial F}{\partial t}-\frac{\partial V}{\partial x} & =0
\end{aligned}
$$

where $V(x, t ; \epsilon)$ is the velocity, $F(x, t ; \epsilon)$ is the displacement gradient, $S(F)$ is the stress, and $\rho\left(x^{*}\right)$ is the density, which is allowed to vary on the fast scale, $x^{*}=x / \epsilon$. (See [2] or [19] for a thorough discussion of elastic waves.) We perturb about the trivial resting state $F=V=0$ by introducing $V=\epsilon u_{1}$ and $F=\epsilon u_{2}$. We also suppose that the relationship between the stress and the displacement gradient is
$S(F)=F+\frac{1}{2} F^{2}$. (We just pick some numbers for simplicity, instead of the more general $S(F)=\alpha F+\beta F^{2}$.) The resulting equations for $\mathbf{u}$ can be written in the form of (4.22) with $B\left(x^{*}\right)=C\left(\mathbf{u}, x^{*}\right)=0$,

$$
A\left(x^{*}\right)=\left[\begin{array}{cc}
0 & -1 / \rho\left(x^{*}\right) \\
-1 & 0
\end{array}\right], \quad \text { and } \quad D\left(\mathbf{u}, x^{*}\right)=\left[\begin{array}{cc}
0 & u_{2} / \rho\left(x^{*}\right) \\
0 & 0
\end{array}\right]
$$

The fact that $B\left(x^{*}\right)=0$ simplifies the analysis of the homogenized equations.
Let us solve an initial-boundary value problem in the domain $x>0$ and $t>0$. We will use $\tilde{x}$ but not $\tilde{t}$ in our asymptotic expansion. It is not hard to show that the long-term evolution equations that arise from removing potentially secular terms in (4.32) are

$$
\begin{align*}
& \underline{w}_{1 \tilde{x}}^{(0)}+\frac{1}{2}\left(\langle\rho\rangle^{-1 / 2}-1\right) \underline{w}_{1 \xi}^{(0)} \underline{w}_{1}^{(0)}=0  \tag{4.33a}\\
& \underline{w}_{2 \tilde{x}}^{(0)}-\frac{1}{2}\left(\langle\rho\rangle^{-1 / 2}+1\right) \underline{w}_{2 \eta}^{(0)} \underline{w}_{2}^{(0)}=0 \tag{4.33b}
\end{align*}
$$

This pair of decoupled, quasilinear, first-order partial differential equations can be solved with requiring any additional initial and boundary information. To see this, we need to examine the solution domains for both equations. The left half of Figure 4.7 shows the solution domain for $\underline{w}_{1}^{(0)}$ in the $\xi-\tilde{x}$ plane and the right half shows the solution domain for $\underline{w}_{2}^{(0)}$ in the $\eta-\tilde{x}$ plane.

Initial conditions for $\underline{w}_{1}^{(0)}$ and $\underline{w}_{2}^{(0)}$ are specified along the rays $\tilde{x}=\epsilon \xi$ and $\tilde{x}=\epsilon \eta$, respectively; the boundary condition is specified as a linear combination of $\underline{w}_{1}^{(0)}$ and $\underline{w}_{2}^{(0)}$ along the rays marked " BC " in Figure 4.7. As long as there are characteristics filling the solution domain for $\underline{w}_{2}^{(0)}$, we should be able to calculate $\underline{w}_{2}^{(0)}$ using the method of characteristics. We can then use the boundary information at $x=\tilde{x}=0$ to obtain a complete solution for $\underline{w}_{1}^{(0)}$.

What are the ways this method could fail? Since the consistency conditions (4.33) are homogeneous, every characteristic is a straight line whose slope is determined



Figure 4.7: Solution domains for the consistency conditions for elastic waves.
solely by the value of $\underline{w}_{i}^{(0)}$ at the point from which the characteristic emanates. Momentarily putting aside the case of shocks, let us consider the possibility that there are regions of the solution domains that are not covered by characteristics. If we disallow discontinuous initial and boundary data, the only way for the characteristics not to fill the solution domains is if the characteristics have slopes between 0 and $\epsilon$. An example of a "bad" characteristic is shown in Figure 4.8.


Figure 4.8: Example of a "bad" characteristic for the second consistency condition for elastic waves.

Calculating the slopes of the characteristics, we obtain the following restriction on the initial conditions for the characteristics to properly fill the solution domain:

$$
\begin{align*}
& \left.\frac{1}{2}\left(\langle\rho\rangle^{-1 / 2}-1\right) \underline{w}_{1}^{(0)}\right|_{t=0}<\frac{1}{\epsilon}  \tag{4.34a}\\
\text { and } \quad & \left.\frac{1}{2}\left(\langle\rho\rangle^{-1 / 2}+1\right) \underline{w}_{2}^{(0)}\right|_{t=0}>-\frac{1}{\epsilon} \tag{4.34b}
\end{align*}
$$

The initial values of $\underline{\mathbf{w}}^{(0)}$ are obtained from $\underline{\mathbf{u}}^{(0)}$ by $\underline{\mathbf{w}}^{(0)}=P^{-1} \underline{\mathbf{u}}^{(0)}$, where

$$
P=\left[\begin{array}{cc}
\langle\rho\rangle^{-1 / 2} & \langle\rho\rangle^{-1 / 2} \\
-1 & 1
\end{array}\right]
$$

Notice that these two inequalities are implicitly satisfied because we perturbed about the steady state $V=F=0$ to obtain this problem. The initial conditions, therefore, cannot not be on the order of $\epsilon^{-1}$.

The only difficulty remaining is that of shocks. Recall from our discussion in Section 3.4 that the formation of shocks necessitates the introduction of $t^{*}=t / \epsilon$ into our asymptotic expansion, putting an analytic solution out of reach. To avoid shocks forming from the outset of the problem, we must choose initial conditions that are continuous, and that match the boundary condition at $x=t=0$ so that there is a single characteristic emanating from the origin in the $\xi$ - $\tilde{x}$ plane.

As an illustration, let us calculate the solution to this problem with the initial conditions

$$
\mathbf{u}(x, 0 ; \epsilon)=\underline{\mathbf{h}}^{(0)}(x)+\mathcal{O}(\epsilon)=\left[\begin{array}{c}
x /(x+1) \\
0
\end{array}\right]+\mathcal{O}(\epsilon)
$$

for $x>0$, and the boundary condition

$$
u_{1}(0, t ; \epsilon)=\frac{t+\frac{1}{2} \sin t}{t+3}+\mathcal{O}(\epsilon)
$$

for $t>0$. We choose $\epsilon=0.1$ and $\rho\left(x^{*}\right)=1+\frac{1}{2} \sin x^{*}$ so $\langle\rho\rangle=1$. The initial conditions satisfy (4.34) and match the boundary condition at $x=t=0$. The characteristic emanating from the origin is the line $x=\langle\rho\rangle^{-1 / 2} t$. Figure 4.9 shows the analytic solution at two different times. The nonlinear effects, though difficult to observe from these graphs, cause slight wave-steepening in the solution. The reason that the solutions for $u_{1}$ and $u_{2}$ appear symmetric is because $\underline{h}_{2}^{(0)}(x)=0$.


Figure 4.9: Analytic solution to the long-term evolution equations (consistency conditions) for elastic waves at two different times.

### 4.2.5 Summary and discussion

We have seen that all of the basic concepts from the linear problem with $B\left(x^{*}\right)=0$ apply equally well to the general nonlinear problem (4.22). These concepts include the relationship between slow and stretched scales and observable long-term phenomena, the construction of consistency conditions, the reason for the missing boundary difficulty, and the recovery of missing boundary information using a reduced set of multiple scales. Generalizing to (4.22), however, does alter the analysis in a few ways.

First, if $B\left(x^{*}\right)$ is nonzero, the operator $\mathcal{L}^{(\mathrm{h})}$ that governs the $x^{*}$-homogenized equations to every order of $\epsilon$ is likely to represent an essentially coupled system of equations. (More precisely, this happens when $P^{-1}\left\langle A^{-1}\right\rangle^{-1}\left\langle A^{-1} B\right\rangle P$ has nonzero offdiagonal terms.) The result of this coupling is that the solution of the homogenized equations involves integrals of Bessel functions, and more importantly, the construction of the consistency conditions from (4.32) is obscured. The consistency conditions have the potential to form a pair of coupled Burgers' equations, whose solution necessitates the recovery of boundary information at $x=0$. In Section 5.1.7, we show how to recover the boundary information for a linear problem with nontrivial $B\left(x^{*}\right)$-this is
sufficient to recover the leading-order boundary information for the general nonlinear problem.

Second, the fact that (4.22) is nonlinear means that we cannot use superposition to partition an initial-boundary value problem into an initial-value problem and a signaling problem. Although it is not any more difficult to recover boundary information for a problem with nontrivial rather than trivial initial conditions, we have seen in Section 4.1.8 that nontrivial initial conditions lead to secular terms in the recovered boundary information. These secular terms limit the region of validity of the computed solution.

Finally, the presence of nonlinearities can actually alleviate the missing boundary difficulty. Whether diffusion is present in the consistency condition (4.32) depends only on the matrix $A\left(x^{*}\right)$, and we have not come across any physically relevant problems that exhibit diffusion. From our analysis of the linear problem with $B\left(x^{*}\right)=0$, we learned that after diffusion, the next possible type of long-term behavior is dispersion, which appears in the $\mathcal{O}\left(\epsilon^{2}\right)$ system of homogenized equations. However, the most significant long-term effects for nonlinear problems manifest themselves at the $\mathcal{O}(\epsilon)$ system of $x^{*}$-homogenized equations. Because these nonlinearities occur one order of $\epsilon$ earlier, and only involve one spatial derivative of the dependent variables, we predict that the consistency conditions for all physically relevant problems will be first-order partial differential equations. As the order of these consistency conditions matches the original partial differential equation, there is a good chance that there no additional boundary conditions are needed to solve the consistency conditions. The initial-boundary value problem of elastic waves in a one-dimensional solid is a perfect example of such a situation.

## Chapter 5

## THE LAPLACE-MULTIPLE-SCALE METHOD

In this chapter, we present an alternative method for handling linear hyperbolic systems with rapid spatial fluctuations using a combination of Laplace transforms and multiple-scales (the Laplace-MS method). We use a Laplace transform in time to turn partial differential equations (PDEs) into ordinary differential equations (ODEs), then apply the multiple-scales method to the resulting ODEs.

We will see that the Laplace-MS method is not useful for finding solutions that account for the long-term behaviors of linear hyperbolic conservation laws, but it will reinforce and confirm all of the findings of the previous chapters in a novel way.

### 5.1 The general linear problem

As the Laplace transform is not suited for solving nonlinear problems, in this chapter we focus on the equation,

$$
\begin{equation*}
\mathbf{u}_{t}+A\left(x^{*}\right) \mathbf{u}_{x}+B\left(x^{*}\right) \mathbf{u}=\mathbf{0} \tag{5.1}
\end{equation*}
$$

which is the most general, linear system of equations that fits the standard form (3.3). The matrices $A\left(x^{*}\right)$ and $B\left(x^{*}\right)$ are $2 \times 2$ matrix functions of the rapid spatial variable $x^{*}=x / \epsilon$, where $0<\epsilon \ll 1$. Our solution procedure involves using the Laplace transform to turn (5.1) into the system of ODEs, then using multiple scales to analyze the ODEs. As in Chapter 4, we include all the necessary spatial scales from the outset of the multiple scale expansion.

A Laplace transform in $t$,

$$
\mathbf{U}(x, s ; \epsilon)=\int_{0}^{\infty} e^{-s t} \mathbf{u}(x, t ; \epsilon) d t
$$

transforms (5.1) into a system of ODEs

$$
\begin{equation*}
\mathbf{U}_{x}+A^{-1}(s l+B) \mathbf{U}=A^{-1} \mathbf{u}(x, 0) \tag{5.2}
\end{equation*}
$$

Now we apply the method of multiple-scales to (5.2). We noted in Chapter 4 that the choice of scales in our multiple-scale expansion is the key to the whole problem. As we intend to analyze (5.1) up to the $\mathcal{O}\left(\epsilon^{2}\right)$ system of equations, we include both $\tilde{x}=\epsilon x$ and $\hat{x}=\epsilon^{2} x$ in our asymptotic expansion:

$$
\begin{equation*}
\mathbf{U}(x, s ; \epsilon)=\underline{\mathbf{U}}^{(0)}(x, \tilde{x}, \hat{x}, s)+\epsilon \mathbf{U}^{(1)}\left(x, x^{*}, \tilde{x}, \hat{x}, s\right)+\cdots . \tag{5.3}
\end{equation*}
$$

Notice that we have already skipped the step of verifying that the leading-order solution is independent of $x^{*}$. (We continue our convention that all $x^{*}$-independent quantities have an underbar.) Derivatives with respect to $x$ become

$$
\frac{d}{d x} \rightarrow \epsilon^{-1} \frac{\partial}{\partial x^{*}}+\frac{\partial}{\partial x}+\epsilon \frac{\partial}{\partial \tilde{x}}+\epsilon^{2} \frac{\partial}{\partial \hat{x}} .
$$

Also, suppose that the initial values for $\mathbf{u}$ can be expanded as

$$
\mathbf{u}(x, 0 ; \epsilon)=\underline{\mathbf{h}}^{(0)}(x)+\epsilon \mathbf{h}^{(1)}\left(x, x^{*}\right)+\cdots,
$$

where the fluctuating part of $\mathbf{h}^{(1)}\left(x, x^{*}\right)$ cannot be specified arbitrarily, as discussed in Section 3.4. We plug all these expansions into (5.2) and collect terms with like powers of $\epsilon$.

### 5.1.1 $\mathcal{O}(1)$ system of ODEs

The $\mathcal{O}(1)$ system of ODEs is

$$
\begin{equation*}
\mathbf{U}_{x^{*}}^{(1)}+\underline{\mathbf{U}}_{x}^{(0)}+A^{-1}(s /+B) \underline{\mathbf{U}}^{(0)}=A^{-1} \underline{\mathbf{h}}^{(0)}(x) \tag{5.4}
\end{equation*}
$$

When we remove $x^{*}$-independent terms that lead to secular growth, we arrive at the $x^{*}$-homogenized system of ODEs

$$
\begin{equation*}
\underline{\mathbf{u}}_{x}^{(0)}+\langle Z\rangle \underline{\mathbf{U}}^{(0)}=\left\langle A^{-1}\right\rangle \underline{\mathbf{h}}^{(0)}(x) \tag{5.5}
\end{equation*}
$$

where $Z=A^{-1}(s I+B)$. The remaining part of the $\mathcal{O}(1)$ system is

$$
\mathbf{U}_{x^{*}}^{(1)}+\{Z\} \underline{\mathbf{U}}^{(0)}=\left\{A^{-1}\right\} \underline{\mathbf{h}}^{(0)}(x)
$$

which we can now integrate to get

$$
\begin{equation*}
\mathbf{U}^{(1)}=-\llbracket Z \rrbracket \underline{\mathbf{U}}^{(0)}+\llbracket A^{-1} \rrbracket \underline{\mathbf{h}}^{(0)}(x)+\underline{\mathbf{U}}^{(1)}(x, \tilde{x}, \hat{x}, s) . \tag{5.6}
\end{equation*}
$$

### 5.1.2 $\mathcal{O}(\epsilon)$ system of ODEs

The $\mathcal{O}(\epsilon)$ system is

$$
\begin{equation*}
\mathbf{U}_{x^{*}}^{(2)}+\mathbf{U}_{x}^{(1)}+\underline{\mathbf{U}}_{\tilde{x}}^{(0)}+Z \mathbf{U}^{(1)}=A^{-1} \mathbf{h}^{(1)}\left(x, x^{*}\right) \tag{5.7}
\end{equation*}
$$

Plug in (5.6) and its derivative to get

$$
\begin{aligned}
\mathbf{U}_{x^{*}}^{(2)}-\llbracket Z \rrbracket \underline{\mathbf{U}}_{x}^{(0)}+\llbracket A^{-1} \rrbracket \underline{\mathbf{h}}_{x}^{(0)} & +\underline{\mathbf{U}}_{x}^{(1)}+\underline{\mathbf{U}}_{\tilde{x}}^{(0)}+Z \underline{\mathbf{U}}^{(1)} \\
& -Z \llbracket Z \rrbracket \underline{\mathbf{U}}^{(0)}+Z \llbracket A^{-1} \rrbracket \underline{\mathbf{h}}^{(0)}=A^{-1} \mathbf{h}^{(1)} .
\end{aligned}
$$

Remove $x^{*}$-independent terms to obtain the $\mathcal{O}(\epsilon)$ homogenized system of ODEs,

$$
\begin{equation*}
\underline{\mathbf{u}}_{x}^{(1)}+\langle Z\rangle \underline{\mathbf{U}}^{(1)}=-\underline{\mathbf{U}}_{\tilde{x}}^{(0)}+\langle Z \llbracket Z \rrbracket\rangle \underline{\mathbf{u}}^{(0)}-\left\langle Z \llbracket A^{-1} \rrbracket\right\rangle \underline{\mathbf{h}}^{(0)}+\left\langle A^{-1} \mathbf{h}^{(1)}\right\rangle . \tag{5.8}
\end{equation*}
$$

We integrate the remaining part of the $\mathcal{O}(\epsilon)$ system to get

$$
\begin{align*}
\mathbf{U}^{(2)}= & \llbracket \llbracket Z \rrbracket \rrbracket \underline{\mathbf{U}}_{x}^{(0)}-\llbracket \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{h}}_{x}^{(0)}-\llbracket Z \rrbracket \underline{\mathbf{U}}^{(1)}+\llbracket Z \llbracket Z \rrbracket \rrbracket \underline{\mathbf{U}}^{(0)}  \tag{5.9}\\
& -\llbracket Z \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{h}}^{(0)}+\llbracket A^{-1} \mathbf{h}^{(1)} \rrbracket+\underline{\mathbf{U}}^{(2)}(x, \tilde{x}, \hat{x}, s) .
\end{align*}
$$

### 5.1.3 $\mathcal{O}\left(\epsilon^{2}\right)$ system of ODEs

The $\mathcal{O}\left(\epsilon^{2}\right)$ system is

$$
\begin{equation*}
\mathbf{U}_{x^{*}}^{(3)}+\mathbf{U}_{x}^{(2)}+\mathbf{U}_{\tilde{x}}^{(1)}+\underline{\mathbf{U}}_{\tilde{x}}^{(0)}+Z \mathbf{U}^{(2)}=A^{-1} \mathbf{h}^{(2)} . \tag{5.10}
\end{equation*}
$$

To clearly see the average and fluctuating parts, we plug in the previous solutions (5.6) and (5.9).

$$
\begin{aligned}
A^{-1} \mathbf{h}^{(2)}= & \mathbf{U}_{x^{*}}^{(3)}-\llbracket Z \rrbracket \underline{\mathbf{U}}_{\tilde{x}}^{(0)}+\underline{\mathbf{U}}_{\tilde{x}}^{(1)}+\llbracket \llbracket Z \rrbracket \rrbracket \underline{\mathbf{U}}_{x x}^{(0)}-\llbracket \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{h}}_{x x}^{(0)}-\llbracket Z \rrbracket \underline{\mathbf{U}}_{x}^{(1)} \\
& +\llbracket Z \llbracket Z \rrbracket \rrbracket \underline{\mathbf{U}}_{x}^{(0)}-\llbracket Z \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{h}}_{x}^{(0)}+\llbracket A^{-1} \mathbf{h}_{x}^{(1)} \rrbracket+\underline{\mathbf{U}}_{x}^{(2)}+\underline{\mathbf{U}}_{\tilde{x}}^{(0)} \\
& +Z \llbracket \llbracket Z \rrbracket \rrbracket \underline{\mathbf{U}}_{x}^{(0)}-Z \llbracket \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{h}}_{x}^{(0)}-Z \llbracket Z \rrbracket \underline{\mathbf{U}}^{(1)} \\
& +Z \llbracket Z \llbracket Z \rrbracket \rrbracket \underline{\mathbf{u}}^{(0)}-Z \llbracket Z \llbracket A^{-1} \rrbracket \rrbracket \underline{\mathbf{h}}^{(0)}+Z \llbracket A^{-1} \mathbf{h}^{(1)} \rrbracket+Z \underline{\mathbf{U}}^{(2)}
\end{aligned}
$$

Remove average terms to obtain the $\mathcal{O}\left(\epsilon^{2}\right)$ homogenized system of ODEs,

$$
\begin{align*}
\underline{\mathbf{U}}_{x}^{(2)}+\langle Z\rangle \underline{\mathbf{U}}^{(2)}= & -\underline{\mathbf{U}}_{\tilde{x}}^{(0)}-\underline{\mathbf{U}}_{\tilde{x}}^{(1)}-\langle Z \llbracket \llbracket Z \rrbracket \rrbracket\rangle \underline{\mathbf{U}}_{x}^{(0)}+\left\langle Z \llbracket \llbracket A^{-1} \rrbracket \rrbracket\right\rangle \underline{\mathbf{h}}_{x}^{(0)} \\
& +\langle Z \llbracket Z \rrbracket\rangle \underline{\mathbf{U}}^{(1)}-\langle Z \llbracket Z \llbracket Z \rrbracket \rrbracket\rangle \underline{\mathbf{U}}^{(0)}+\left\langle Z \llbracket Z \llbracket A^{-1} \rrbracket \rrbracket\right\rangle \underline{\mathbf{h}}^{(0)}  \tag{5.11}\\
& -\left\langle Z \llbracket A^{-1} \mathbf{h}^{(1)} \rrbracket\right\rangle+\left\langle A^{-1} \mathbf{h}^{(2)}\right\rangle .
\end{align*}
$$

The remaining part of the $\mathcal{O}\left(\epsilon^{2}\right)$ system can be integrated to obtain

$$
\begin{align*}
\mathbf{U}^{(3)}= & \llbracket A^{-1} \mathbf{h}^{(2)} \rrbracket+\llbracket \llbracket Z \rrbracket \rrbracket \underline{\mathbf{U}}_{\tilde{x}}^{(0)}-\llbracket \llbracket \llbracket Z \rrbracket \rrbracket \rrbracket \underline{\mathbf{U}}_{x x}^{(0)}+\llbracket \llbracket \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \underline{\mathbf{h}}_{x x}^{(0)}+\llbracket \llbracket Z \rrbracket \rrbracket \underline{\mathbf{U}}_{x}^{(1)} \\
& -\llbracket \llbracket Z \llbracket \rrbracket \rrbracket \rrbracket \underline{\mathbf{U}}_{x}^{(0)}+\llbracket \llbracket Z \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \underline{\mathbf{h}}_{x}^{(0)}-\llbracket \llbracket A^{-1} \mathbf{h}_{x}^{(1)} \rrbracket \rrbracket-\llbracket Z \rrbracket \underline{\mathbf{u}}^{(2)} \\
& -\llbracket Z \llbracket \llbracket Z \rrbracket \rrbracket \rrbracket \underline{\mathbf{u}}_{x}^{(0)}+\llbracket Z \llbracket \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \underline{\mathbf{h}}_{x}^{(0)}+\llbracket Z \llbracket Z \rrbracket \rrbracket \underline{\mathbf{u}}^{(1)} \\
& -\llbracket Z \llbracket Z \llbracket Z \rrbracket \rrbracket \rrbracket \underline{\mathbf{u}}^{(0)}+\llbracket Z \llbracket Z \llbracket A^{-1} \rrbracket \rrbracket \rrbracket \underline{\mathbf{h}}^{(0)}-\llbracket Z \llbracket A^{-1} \mathbf{h}^{(1)} \rrbracket \rrbracket+\underline{\mathbf{U}}^{(3)}(x, \tilde{x}, \hat{x}, s) . \tag{5.12}
\end{align*}
$$

### 5.1.4 Choosing initial conditions for $t^{*}$-independent solutions

Armed with expressions for $\underline{\mathbf{U}}^{(0)}, \mathbf{U}^{(1)}$, etc., we can now address the question of what initial conditions give rise to solutions that are independent of $t^{*}=t / \epsilon$. We cannot
perform the natural substitution $t=0$ because the solution is still in the Laplace domain-it needs to be inverted first.

First, we observe that if $\underline{\mathbf{U}}^{(0)}$ is independent of $x^{*}$, then $\underline{\mathbf{h}}^{(0)}=\underline{\mathbf{h}}^{(0)}(x)$ should also be independent of $x^{*}$.

To see what happens at the next order, we rewrite (5.6) using the definition $Z=A^{-1}(s I+B)$ to get

$$
\mathbf{U}^{(1)}=-\llbracket A^{-1} \rrbracket\left(s \underline{\mathbf{U}}^{(0)}-\underline{\mathbf{h}}^{(0)}(x)\right)-\llbracket A^{-1} B \rrbracket \underline{\mathbf{U}}^{(0)}+\underline{\mathbf{U}}^{(1)} .
$$

Recognizing that the Laplace transform of $f^{\prime}(t)$ is $s F(s)-f(0)$, when we apply an inverse Laplace transform, we get

$$
\mathbf{u}^{(1)}=-\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t}^{(0)}-\llbracket A^{-1} B \rrbracket \underline{\mathbf{u}}^{(0)}+\underline{\mathbf{u}}^{(1)}
$$

so

$$
\begin{equation*}
\mathbf{h}^{(1)}\left(x, x^{*}\right)=-\left.\llbracket A^{-1} \rrbracket \underline{\mathbf{u}}_{t}^{(0)}\right|_{t=0}-\llbracket A^{-1} B \rrbracket \underline{\mathbf{h}}^{(0)}+\underline{\mathbf{h}}^{(1)}(x) . \tag{5.13}
\end{equation*}
$$

The average part of the $\mathcal{O}(\epsilon)$ initial conditions, $\underline{\mathbf{h}}^{(1)}(x)$, may be chosen arbitrarily but the fluctuating part is fixed by our assumption that the solution is independent of $t^{*}=t / \epsilon$. Incidentally, (5.13) is also obtained using the standard multiple-scale technique. This analysis may be continued for higher orders of $\epsilon$.

### 5.1.5 Solving the homogenized ODEs for the signaling problem

In this section, we solve the $x^{*}$-homogenized ordinary differential equations that we generated in the analysis above, specializing to the signaling problem in the semiinfinite domain $x>0$ and $t>0$. We don't lose any generality by assuming that the initial conditions in this problem are zero because we can use superposition to partition a problem with both initial and boundary conditions into two separate problems: an initial-value problem on the infinite domain $-\infty<x<\infty$ and a signaling problem in the semi-infinite domain.

To fix ideas, suppose that $u_{1}(0, t ; \epsilon)=g(t ; \epsilon)$ for $t>0$ is the specified boundary condition ${ }^{1}$ and $\mathbf{u}(x, 0 ; \epsilon)=\mathbf{h}(x ; \epsilon)$ for $x>0$ is the initial condition. Let us write $\mathbf{u}(x, t ; \epsilon)=\mathbf{v}(x, t ; \epsilon)+\mathbf{w}(x, t ; \epsilon)$, where $\mathbf{v}$ and $\mathbf{w}$ both satisfy (5.1), but the spatial domain for $\mathbf{v}$ is $-\infty<x<\infty$ and the domain for $\mathbf{w}$ is $x>0$. We extend the initial conditions $\mathbf{h}(x ; \epsilon)$ however we like, and enforce $\mathbf{v}(x, 0 ; \epsilon)=\mathbf{h}(x ; \epsilon)$. Once we obtain the solution to the initial value problem (addressed in Section 4.1.6 and [19]), the only remaining task is to solve the signaling problem (zero initial conditions) with the boundary condition $w_{1}(0, t ; \epsilon)=g(t ; \epsilon)-v_{1}(0, t ; \epsilon)$ for $t>0$. Since (5.1) is a linear equation, we can add $\mathbf{v}$ and $\mathbf{w}$ to obtain the solution to the general problem with both initial and boundary conditions. It doesn't matter how the initial conditions $\mathbf{h}(x ; \epsilon)$ are extended for negative $x$ because these values will be subtracted again from the boundary condition for $w_{1}(0, t ; \epsilon)$. This partitioning allows us to focus on the signaling problem for (5.1) in this section. The analysis for nonzero initial conditions is messier because it involves solving inhomogeneous ordinary differential equations, but the concepts are the same.

Now let us solve (5.1), supposing that $u_{1}(0, t ; \epsilon)=g(t ; \epsilon)$ for $t>0$ is our boundary condition and $\mathbf{u}(x, 0 ; \epsilon)=\mathbf{0}$ for $x>0$ is our initial condition. With the initial conditions eliminated, the $\mathcal{O}(1)$ homogenized system of ODEs, equation (5.5), becomes

$$
\underline{\mathbf{u}}_{x}^{(0)}+\langle Z\rangle \underline{\mathbf{U}}^{(0)}=\mathbf{0}
$$

where $Z=A^{-1}(s l+B)$. The solution to this homogeneous system of equations is

$$
\begin{equation*}
\underline{\mathbf{U}}^{(0)}=e^{-\langle Z\rangle x} \underline{\mathbf{V}}^{(0)}(\tilde{x}, \hat{x}, s), \tag{5.14}
\end{equation*}
$$

where $\underline{\mathbf{V}}^{(0)}$ is a constant of integration. We do not know the $\tilde{x}$ - or $\hat{x}$-dependence of $\underline{\mathbf{V}}^{(0)}$ until we consider the equations arising at higher orders of $\epsilon$, but there is one

[^4]important difficulty: we only know $u_{1}(0, t ; \epsilon)$, so the boundary condition alone is not enough to solve the problem. The answer is to also enforce whatever conditions are necessary for the solution to be Laplace-invertible. Here's how.

With a little bit of algebra, one can write

$$
\begin{equation*}
e^{\langle Z\rangle x}=J_{2} e^{\lambda_{1} x}-J_{1} e^{\lambda_{2} x} \tag{5.15}
\end{equation*}
$$

where $\lambda_{1}$ and $\lambda_{2}$ are the eigenvalues of $\langle Z\rangle$, and

$$
L_{i}=\frac{1}{\mu}\left(\langle Z\rangle-\lambda_{i} l\right)
$$

for $i=1$, 2. Since $Z=A^{-1}(s I+B)$, the eigenvalues $\lambda_{i}$ are actually functions of $s$ :

$$
\begin{equation*}
\lambda_{1,2}=\frac{\operatorname{tr}\langle Z\rangle \pm \mu}{2} \tag{5.16}
\end{equation*}
$$

where $\mu=\lambda_{1}-\lambda_{2}=\sqrt{\operatorname{tr}^{2}\langle Z\rangle-4 \operatorname{det}\langle Z\rangle}$. So now we can see that in the expression

$$
\underline{\mathbf{U}}^{(0)}=e^{-\langle Z\rangle x} \underline{\mathbf{V}}^{(0)}(\tilde{x}, \hat{x}, s)=\left[J_{2} e^{-\lambda_{1} x}-J_{1} e^{-\lambda_{2} x}\right] \underline{\mathbf{V}}^{(0)}(\hat{x}, s)
$$

all terms involving $e^{-\lambda_{2} x}$ are not Laplace-invertible because $\lambda_{2}$ has negative real part ${ }^{2}$. Therefore, to eliminate these terms, we need $J_{1} \underline{\mathbf{V}}^{(0)}=\mathbf{0}$, which is equivalent to $\underline{\mathbf{V}}^{(0)}$ being an eigenvector associated with $\lambda_{1}$. The boundary information is the other piece of information that will allow us to determine $\underline{\mathbf{V}}^{(0)}$ completely.

The $\mathcal{O}(\epsilon)$ homogenized system of ODEs, equation (5.8), with zero initial conditions, is

$$
\underline{\mathbf{U}}_{x}^{(1)}+\underline{\mathbf{U}}_{\tilde{x}}^{(0)}+\langle Z\rangle \underline{\mathbf{U}}^{(1)}-\langle Z \llbracket Z \rrbracket\rangle \underline{\mathbf{U}}^{(0)}=\mathbf{0} .
$$

To solve this equation, we make the substitution $\underline{\mathbf{U}}^{(1)}=e^{-\langle Z\rangle \times} \underline{\mathbf{M}}^{(1)}$ to obtain

$$
\begin{equation*}
\underline{\mathbf{M}}_{x}^{(1)}=-\underline{\mathbf{V}}_{\tilde{x}}^{(0)}+e^{\langle Z\rangle \times}\langle Z \llbracket Z \rrbracket\rangle e^{-\langle Z\rangle \times} \underline{\mathbf{V}}^{(0)} . \tag{5.17}
\end{equation*}
$$

[^5]Before, we integrate with respect to $x$, we first need to remove terms that are independent of $x$ (like $\underline{\mathbf{V}}^{(0)}$ ), so we do not get secular terms.

Using (5.15), we calculate

$$
\begin{align*}
e^{\langle Z\rangle x}\langle Z \llbracket Z \rrbracket\rangle e^{-\langle Z\rangle x}= & J_{1}\langle Z \llbracket Z \rrbracket\rangle J_{1}+J_{2}\langle Z \llbracket Z \rrbracket\rangle J_{2}  \tag{5.18}\\
& -J_{2}\langle Z \llbracket Z \rrbracket\rangle J_{1} e^{\left(\lambda_{1}-\lambda_{2}\right) x}-J_{1}\langle Z \llbracket Z \rrbracket\rangle J_{2} e^{\left(\lambda_{2}-\lambda_{1}\right) x} .
\end{align*}
$$

It is the matrix $C^{(0)}=J_{1}\langle Z \llbracket Z \rrbracket\rangle J_{1}+J_{2}\langle Z \llbracket Z \rrbracket\rangle J_{2}$ that we are concerned about, because it is independent of $x$ and leads to secular terms. The removal of these potentially secular terms in (5.17) produces the ordinary differential equation

$$
\underline{\mathbf{V}}_{\tilde{x}}^{(0)}=C^{(0)} \underline{\mathbf{V}}^{(0)}
$$

which has the solution

$$
\begin{equation*}
\underline{\mathbf{V}}^{(0)}=\exp \left(C^{(0)} \tilde{x}\right) \underline{\mathbf{W}}^{(0)}(\hat{x}, s) \tag{5.19}
\end{equation*}
$$

The difficulty with (5.19) is that, in general, it is not Laplace invertible. The reason for this is that the entries of $\langle Z\rangle$ are linear in $s$, so the eigenvalues involve linear functions and square roots of quadratic functions in s. Using the definitions of $L_{i}$, we see that calculating $C^{(0)}$ involves squaring some eigenvalues, which will lead to quadratic functions in $s$. It is not possible to find the Laplace inverse of a function containing $\exp \left(c s^{2}\right)$ for any constant $c$ (see Chapter 28 of [12]). In other words, there is no function of $t$ that has $\exp \left(c s^{2}\right)$ as its Laplace transform. The presence of noninvertible terms is why the Laplace-MS method is unable to construct solutions that account for the long-term effects of the fluctuations in $A\left(x^{*}\right)$ and $B\left(x^{*}\right)$.

### 5.1.6 Signaling problem for the linear wave equation

As an illustration, let us solve the homogenized ODEs that arise for the linear wave equation,

$$
\begin{equation*}
\rho\left(x^{*}\right) w_{t t}-\left(k\left(x^{*}\right) w_{x}\right)_{x}=0, \tag{5.20}
\end{equation*}
$$

where $w(x, t ; \epsilon), \rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$ are the displacement, density and bulk modulus, respectively. We assume that $\rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$ are positive functions. As in Section 4.1, we rewrite this second-order equation as a system of first-order PDEs in the form of (5.1) with

$$
A\left(x^{*}\right)=\left[\begin{array}{cc}
0 & k\left(x^{*}\right) \\
1 / \rho\left(x^{*}\right) & 0
\end{array}\right]
$$

and $B\left(x^{*}\right)=0$.
We begin by performing some matrix calculations. The matrix $Z$ is

$$
Z=A^{-1}(s l+B)=s\left[\begin{array}{cc}
0 & \rho\left(x^{*}\right) \\
1 / k\left(x^{*}\right) & 0
\end{array}\right]
$$

and the eigenvalues of $\langle Z\rangle$ are $\lambda_{1}=s\langle\rho\rangle\left\langle k^{-1}\right\rangle$ and $\lambda_{2}=-s\langle\rho\rangle\left\langle k^{-1}\right\rangle$. The matrix exponential $e^{\langle Z\rangle x}$ can be expressed as

$$
e^{\langle Z\rangle x}=J_{2} e^{\lambda_{1} x}-J_{1} e^{\lambda_{2} x}
$$

where

$$
J_{1}=\frac{1}{2}\left[\begin{array}{cc}
-1 & \frac{\langle\rho\rangle^{1 / 2}}{\left\langle k^{-1}\right\rangle^{1 / 2}} \\
\frac{\left\langle k^{-1}\right\rangle^{1 / 2}}{\langle\rho\rangle^{1 / 2}} & -1
\end{array}\right] \quad \text { and } \quad J_{2}=\frac{1}{2}\left[\begin{array}{cc}
1 & \frac{\langle\rho\rangle^{1 / 2}}{\left\langle k^{-1}\right\rangle^{1 / 2}} \\
\frac{\left\langle k^{-1}\right\rangle^{1 / 2}}{\langle\rho\rangle^{1 / 2}} & 1
\end{array}\right]
$$

Also,

$$
\langle Z \llbracket Z \rrbracket\rangle=s^{2}\left[\begin{array}{cc}
\left\langle k^{-1} \llbracket \rho \rrbracket\right\rangle & 0 \\
0 & \left\langle\rho \llbracket k^{-1} \rrbracket\right\rangle
\end{array}\right]=s^{2}\left\langle k^{-1} \llbracket \rho \rrbracket\right\rangle\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right] .
$$

(We have used the fact that $\left\langle k^{-1} \llbracket \rho \rrbracket\right\rangle=-\left\langle\rho \llbracket k^{-1} \rrbracket\right\rangle$, which we prove in Appendix A.)
A simple calculation will show that regardless of $\rho\left(x^{*}\right)$ and $k\left(x^{*}\right)$, the product $L_{i}\langle Z \llbracket Z \rrbracket\rangle L_{i}=0$ for $i=1,2$, so that $C^{(0)}=J_{1}\langle Z \llbracket Z \rrbracket\rangle J_{1}+J_{2}\langle Z \llbracket Z \rrbracket\rangle J_{2}=0$. Equation (5.19) tells us the significance of this finding: that $\tilde{x}$ is not needed in our asymptotic expansion. This fits with our analysis in Chapter 4 that there is no diffusion
on the $\tilde{t}=\epsilon t$ or $\tilde{x}$ scale. Now, we examine the $x^{*}$-homogenized system of ODEs at each order of $\epsilon$, neglecting all initial conditions and derivatives with respect to $\tilde{x}$.

The $\mathcal{O}(1)$ homogenized system of ODEs,

$$
\begin{equation*}
\underline{\mathbf{u}}_{x}^{(0)}+\langle Z\rangle \underline{\mathbf{U}}^{(0)}=\mathbf{0}, \tag{5.21}
\end{equation*}
$$

has the solution

$$
\underline{\mathbf{U}}^{(0)}=e^{-\langle Z\rangle x} \underline{\mathbf{W}}^{(0)}(\hat{x}, s)=\left[J_{2} e^{-\lambda_{1} x}-J_{1} e^{-\lambda_{2} x}\right] \underline{\mathbf{W}}^{(0)}(\hat{x}, s)
$$

Because $-\lambda_{2}=s\langle\rho\rangle\left\langle k^{-1}\right\rangle$ and $\langle\rho\rangle\left\langle k^{-1}\right\rangle>0$, we must eliminate all occurrences of $e^{-\lambda_{2} x}$ in $\underline{\mathbf{U}}^{(0)}$ if we want a solution that is Laplace-invertible. (Also, we want the solution to remain bounded as $x \rightarrow \infty$.) Therefore, we must enforce $J_{1} \underline{\mathbf{W}}^{(0)}=\mathbf{0}$, making $\underline{\mathbf{W}}^{(0)}$ an eigenvector associated with $\lambda_{1}$. Now $\underline{\mathbf{U}}^{(0)}=J_{2} e^{-\lambda_{1} \times} \underline{\mathbf{W}}^{(0)}(\hat{x}, s)$.

Neglecting $\tilde{x}$ derivatives, the $\mathcal{O}(\epsilon)$ system of $x^{*}$-homogenized ODEs is

$$
\underline{\mathbf{U}}_{x}^{(1)}+\langle Z\rangle \underline{\mathbf{u}}^{(1)}=\langle Z \llbracket Z \rrbracket\rangle \underline{\mathbf{u}}^{(0)} .
$$

As before, we make the substitution $\underline{\mathbf{U}}^{(1)}=e^{-\langle Z\rangle x} \underline{\mathbf{M}}^{(1)}(x, \hat{x}, s)$ to obtain

$$
\underline{\mathbf{M}}_{x}^{(1)}=e^{\langle Z\rangle x}\langle Z \llbracket Z \rrbracket\rangle e^{-\langle Z\rangle x} \underline{\mathbf{w}}^{(0)}=-J_{1}\langle Z \llbracket Z \rrbracket\rangle J_{2} e^{\left(\lambda_{2}-\lambda_{1}\right) x} \underline{\mathbf{W}}^{(0)} .
$$

(We have used (5.18) and the fact that $J_{1} \underline{\mathbf{W}}^{(0)}=\mathbf{0}$.) Notice that there are indeed no terms independent of $x$, so we can integrate to obtain

$$
\underline{\mathbf{M}}^{(1)}(x, \hat{x}, s)=\frac{1}{\lambda_{1}-\lambda_{2}} J_{1}\langle Z \llbracket Z \rrbracket\rangle J_{2} e^{\left(\lambda_{2}-\lambda_{1}\right) x} \underline{\mathbf{W}}^{(0)}+\underline{\mathbf{W}}^{(1)}(\hat{x}, s)
$$

Because $\underline{\mathbf{U}}^{(1)}=e^{-\langle Z\rangle x} \underline{\mathbf{M}}^{(1)}(x, \hat{x}, s)$, we need to enforce $J_{1} \underline{\mathbf{V}}^{(1)}=\mathbf{0}$, as we did for $\underline{\mathbf{V}}^{(0)}$, to ensure that $\underline{\mathbf{U}}^{(1)}$ is Laplace-invertible.

Finally, we turn to the $\mathcal{O}\left(\epsilon^{2}\right)$ homogenized system of equations (5.11) and set all initial conditions to zero:

$$
\begin{aligned}
\underline{\mathbf{U}}_{x}^{(2)}+\langle Z\rangle \underline{\mathbf{U}}^{(2)} & =-\underline{\mathbf{u}}_{\hat{x}}^{(0)}-\langle Z \llbracket \llbracket Z \rrbracket \rrbracket\rangle \underline{\mathbf{u}}_{x}^{(0)}+\langle Z \llbracket Z \rrbracket\rangle \underline{\mathbf{U}}^{(1)}-\langle Z \llbracket Z \llbracket Z \rrbracket \rrbracket\rangle \underline{\mathbf{U}}^{(0)} \\
& =-\underline{\mathbf{u}}_{\hat{x}}^{(0)}+(\langle Z \llbracket \llbracket Z \rrbracket \rrbracket\rangle\langle Z\rangle-\langle Z \llbracket Z \llbracket Z \rrbracket \rrbracket\rangle) \underline{\mathbf{U}}^{(0)}+\langle Z \llbracket Z \rrbracket\rangle \underline{\mathbf{u}}^{(1)} .
\end{aligned}
$$

(The second equality above comes applying (5.21).) Let $\underline{\mathbf{U}}^{(2)}(x, \hat{x}, s)=e^{-\langle Z\rangle \times} \underline{\mathbf{M}}^{(2)}(\hat{x}, s)$ to obtain

$$
\begin{aligned}
\underline{\mathbf{M}}_{x}^{(2)}= & -\underline{\mathbf{W}}_{\hat{x}}^{(0)}+e^{\langle Z\rangle x}(\langle Z \llbracket \llbracket Z \rrbracket \rrbracket\rangle\langle Z\rangle-\langle Z \llbracket Z \llbracket Z \rrbracket \rrbracket\rangle) e^{-\langle Z\rangle x} \underline{\mathbf{W}}^{(0)} \\
& +e^{\langle Z\rangle \times}\langle Z \llbracket Z \rrbracket\rangle e^{-\langle Z\rangle x} \underline{\mathbf{M}}^{(1)} .
\end{aligned}
$$

We must remove terms independent of $x$ to avoid secular terms. This requires

$$
\underline{\mathbf{w}}_{\widehat{\chi}}^{(0)}=\left(C^{(1)}-\frac{1}{\lambda_{1}-\lambda_{2}} J_{2}\langle Z \llbracket Z \rrbracket\rangle\left(J_{1}\right)^{2}\langle Z \llbracket Z \rrbracket\rangle J_{2}\right) \underline{\mathbf{w}}^{(0)},
$$

where

$$
\begin{aligned}
C^{(1)} & =\sum_{i=1}^{2} J_{i}(\langle Z \llbracket \llbracket Z \rrbracket \rrbracket\rangle\langle Z\rangle-\langle Z \llbracket Z \llbracket Z \rrbracket \rrbracket\rangle) J_{i} \\
& =s^{3}\left[\begin{array}{cc}
0 & \mu+\sigma \frac{\langle\rho\rangle}{\left\langle k^{-1}\right\rangle} \\
\sigma+\mu \frac{\left\langle k^{-1}\right\rangle}{\langle\rho\rangle}
\end{array}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
\mu & =\left\langle\rho \llbracket k^{-1} \rrbracket^{2}\right\rangle-\left\langle k^{-1}\right\rangle\left\langle\llbracket k^{-1} \rrbracket \llbracket \rho \rrbracket\right\rangle \\
\sigma & =\left\langle k^{-1} \llbracket \rho \rrbracket^{2}\right\rangle-\langle\rho\rangle\left\langle\llbracket k^{-1} \rrbracket \llbracket \rho \rrbracket\right\rangle .
\end{aligned}
$$

For the particular choice

$$
\begin{aligned}
& \rho\left(x^{*}\right)=1+a_{1} \cos \left(\pi x^{*}\right)+a_{2} \sin \left(\pi x^{*}\right) \quad \text { and } \\
& k\left(x^{*}\right)=\frac{1}{1+b_{1} \cos \left(\pi x^{*}\right)+b_{2} \sin \left(\pi x^{*}\right)}
\end{aligned}
$$

it turns out that $J_{1} \underline{\mathbf{W}}^{(0)}=\mathbf{0}$ implies $\underline{W}_{1}^{(0)}=\underline{W}_{2}^{(0)}$. In addition,

$$
C^{(1)}=s^{3} \frac{\left(a_{1}-b_{1}\right)^{2}+\left(a_{2}-b_{2}\right)^{2}}{4 \pi^{2}}\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right],
$$

and $\underline{\mathbf{W}}^{(0)}$ is in the null space of $J_{2}\langle Z \llbracket Z \rrbracket\rangle\left(J_{1}\right)^{2}\langle Z \llbracket Z \rrbracket\rangle J_{2}$. Therefore, $\underline{W}_{1}^{(0)}$ satisfies

$$
\underline{W}_{1 \hat{x}}^{(0)}=s^{3} \frac{\left(a_{1}-b_{1}\right)^{2}+\left(a_{2}-b_{2}\right)^{2}}{4 \pi^{2}} \underline{W}_{1}^{(0)}
$$

which has the solution

$$
\begin{equation*}
\underline{W}_{1 \hat{x}}^{(0)}=\exp \left(s^{3} \hat{x} \frac{\left(a_{1}-b_{1}\right)^{2}+\left(a_{2}-b_{2}\right)^{2}}{4 \pi^{2}}\right) G^{(0)}(s) \tag{5.22}
\end{equation*}
$$

where $G^{(0)}(s)$ is the Laplace transform of $g^{(0)}(t)$. Now, the inverse Laplace transform of the function $\exp \left(s^{3}\right)$ does not exist [12], and only if $a_{1}=b_{1}$ and $a_{2}=b_{2}$, or equivalently, $\rho\left(x^{*}\right)=1 / k\left(x^{*}\right)$, will this non-invertible term go away. However, when $\rho\left(x^{*}\right)=1 / k\left(x^{*}\right)$, multiple-scale analysis is not necessary because the linear wave equation admits an exact solution. The presence of non-invertible terms means that the Laplace-MS method cannot be used to generate solutions that account for behaviors on the $\hat{x}$ scale. If we were trying to use the Laplace transform to solve a nonphysical problem, the presence of non-invertible terms would not be surprising; it is therefore curious these terms appear in the solution to the linear wave equation.

Replacing the non-invertible exponential term in (5.22) with a truncated version of its power series expansion seems promising at first. After all, truncating the power series produces a polynomial in $s$, which is perfectly Laplace-invertible. However, the truncated power series will contain secular terms proportional to $\hat{x}$ so this remedy is really no better than the solution obtained via a multiple-scale analysis involving $x^{*}$ and $x$ only (neglecting $\hat{x}$ ).

There is an interesting connection between the appearance of these non-invertible terms and the multiple-scale analysis in Chapter 4. Performing the usual multiplescale analysis with stretched spatial scales instead of slow temporal scales, produces long-term evolution equations with three $t$-derivatives instead of three $x$-derivatives (refer to equation (4.13)). A Laplace transform with respect to $t$ would turn this strange PDE into a first-order ODE in $x$, and we would also obtain non-invertible terms like $\exp \left(s^{3}\right)$. This fact suggests that the presence of non-invertible terms in the Laplace-MS method is the manifestation of the missing boundary condition difficulty in the usual multiple-scale method. In all the examples that we have studied, the power
of $s$ in the non-invertible term matches the order of the derivative in the long-term evolution equation that causes the missing boundary condition difficulty.

### 5.1.7 Recovering boundary conditions

Even though the Laplace-MS method cannot be used to find solutions to (5.1) that take into account the long-term effects of the fluctuations in $A\left(x^{*}\right)$ and $B\left(x^{*}\right)$, it can be used to recover boundary conditions like the reduced multiple-scale analysis in Section 4.1.8. For example, notice that the noninvertible term in (5.22) goes away when we substitute $x=\hat{x}=0$. Once the troublesome term is gone, we can apply the inverse Laplace transform to recover the values of $\mathbf{u}$ at $x=0$.

Rather than keeping the stretched spatial scales in the Laplace-MS analysis and hoping that the noninvertible terms will go away when we substitute $x=0$, a simpler approach is to give up using stretched spatial scales in our asymptotic expansion for $\mathbf{U}(x, s ; \epsilon)$. Without these stretched scales, we cannot avoid obtaining secular terms proportional to $x$ in our solution. These secular terms limit the region of validity of the solution, but as we are only concerned with the values at $x=0$, they don't matter.

Let us calculate the recovered boundary information for (5.1) with zero initial conditions and

$$
u_{1}(0, t ; \epsilon)=g(t ; \epsilon)=g^{(0)}(t)+\mathcal{O}(\epsilon)
$$

We re-expand the solution as

$$
\mathbf{U}(x, s ; \epsilon)=\underline{\mathbf{R}}^{(0)}(x, s)+\mathbf{R}^{(1)}\left(x^{*}, x, s\right)+\mathcal{O}\left(\epsilon^{2}\right)
$$

using $\mathbf{R}^{(i)}$ instead of $\mathbf{U}^{(i)}$ to avoid confusion with the previous multiple-scale expansion (5.3). We follow the definitions and method outlined in Section 5.1 .5 to obtain

$$
\underline{\mathbf{R}}^{(0)}(x, s)=e^{-\langle Z\rangle x} \underline{\mathbf{R}}^{(0)}(0, s)=\left[J_{2} e^{-\lambda_{1} x}-J_{1} e^{-\lambda_{2} x}\right] \underline{\mathbf{R}}^{(0)}(0, s) .
$$

As we discussed earlier, we do not want $\exp \left(-\lambda_{2} x\right)$ in the solution because it is not Laplace-invertible. Therefore, we should restrict $J_{1} \underline{\mathbf{R}}^{(0)}(0, s)=\mathbf{0}$. The first
component of $\underline{\mathbf{R}}^{(0)}(0, s)$ is equal to $G^{(0)}(s)$, the Laplace transform of the boundary condition, so,

$$
\underline{\mathbf{R}}^{(0)}(0, s)=\left[\begin{array}{c}
1 \\
-j_{11} / j_{12}
\end{array}\right] G^{(0)}(s)
$$

where $j_{m n}$ refers to the $m-n$ entry of $J_{1}$. This procedure can be repeated to obtain higher-order corrections to the recovered boundary information. It can also be used to recover boundary information for problems with nontrivial initial conditions, although each system of homogenized ODEs will be inhomogeneous. We have developed Mathematica notebooks capable of performing these tedious calculations to any desired order of $\epsilon$. Once $\mathbf{U}(0, s ; \epsilon)$ is calculated to the desired accuracy, we just need to apply the Laplace inversion to obtain the recovered boundary information in physical variables.

For example, let us recompute the missing boundary information for the two examples presented in Section 4.1.8. The boundary information will be calculated for the linear wave equation with

$$
\begin{aligned}
& \rho\left(x^{*}\right)=1+0.5 \cos \left(\pi x^{*}\right)-0.3 \sin \left(\pi x^{*}\right) \quad \text { and } \\
& k\left(x^{*}\right)=\frac{1}{1-0.1 \cos \left(\pi x^{*}\right)+0.25 \sin \left(\pi x^{*}\right)}
\end{aligned}
$$

and the given boundary condition $u_{1}(0, t ; \epsilon)=1-\cos t+\mathcal{O}(\epsilon)$. For the signaling problem (no initial conditions), the Laplace-MS method predicts that

$$
\begin{equation*}
u_{2}(0, t ; \epsilon)=1-\cos t-\epsilon \frac{201 \sin t}{400 \pi}+\epsilon^{2} \frac{399361 \cos t}{320000 \pi^{2}}+\mathcal{O}\left(\epsilon^{3}\right) \tag{5.23}
\end{equation*}
$$

This recovered boundary information matches our previous computation (4.18) using multiple scales alone.

In the presence of the initial conditions

$$
\mathbf{u}(x, 0 ; \epsilon)=\left[\begin{array}{c}
\sin (\pi x) \\
0
\end{array}\right]+\mathcal{O}(\epsilon)
$$

the recovered boundary information via the Laplace-MS method is

$$
\begin{align*}
u_{2}(0, t ; \epsilon)= & 1-\cos (t)-\sin (\pi t)+\epsilon\left(\frac{\cos (\pi t)}{4}-\frac{201 \sin t}{400 \pi}\right) \\
& +\epsilon^{2}\left(\frac{52639 \pi t \cos (\pi t)}{320000}+\frac{399361 \cos t+209161 \pi^{2} \sin (\pi t)}{320000 \pi^{2}}\right)+\mathcal{O}\left(\epsilon^{3}\right) \tag{5.24}
\end{align*}
$$

Again, equation (5.24) matches (4.17), the recovered boundary information for the same problem from Section 4.1.8.

We end this section by demonstrating the recovery of boundary information for a problem in which $B\left(x^{*}\right) \neq 0$. Consider Maxwell's equations specialized for plane polarized waves, propagating in a one-dimensional medium. If the current density is linearly related to the electric field, the governing equations may be expressed in the form (5.1), where

$$
A\left(x^{*}\right)=\left[\begin{array}{cc}
0 & \varepsilon\left(x^{*}\right)^{-1} \\
\mu\left(x^{*}\right)^{-1} & 0
\end{array}\right] \quad \text { and } \quad B\left(x^{*}\right)=\left[\begin{array}{cc}
\sigma\left(x^{*}\right) & 0 \\
0 & 0
\end{array}\right] .
$$

Here, $\varepsilon\left(x^{*}\right)$ (different from $\epsilon$ ), $\mu\left(x^{*}\right)$, and $\sigma\left(x^{*}\right)$ are the dielectric constant, permeability, and conductivity, respectively. The first dependent variable, $u_{1}(x, t ; \epsilon)$, is the electric field, and the second is the magnetic field. All variables have been appropriately nondimensionalized. (These equations are derived in [19] for a nonlinear medium.)

Suppose we have a signaling problem in which $u_{1}(0, t ; \epsilon)=g(t ; \epsilon)=g^{(0)}(t)+\mathcal{O}(\epsilon)$ and $\underline{\mathbf{u}}^{(0)}(x, 0)=\mathbf{0}$. The matrix $Z=A^{-1}(s I+B)$ has the average value

$$
\langle Z\rangle=\left[\begin{array}{cc}
0 & \langle\mu\rangle s \\
\langle\varepsilon\rangle s+\langle\varepsilon \sigma\rangle & 0
\end{array}\right]
$$

After some algebra, we find that

$$
\underline{U}_{2}^{(0)}(0, s)=\left(\frac{s\langle\varepsilon\rangle+\langle\varepsilon \sigma\rangle}{s\langle\mu\rangle}\right)^{1 / 2} G^{(0)}(s)
$$

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Applying an inverse Laplace transform, the recovered boundary information is

$$
\underline{u}_{2}^{(0)}(0, t)=\alpha \sqrt{\frac{\langle\varepsilon\rangle}{\langle\mu\rangle}} \int_{0}^{t} e^{-\alpha \tau}\left(\mathcal{I}_{0}(\alpha \tau)+\mathcal{I}_{1}(\alpha \tau)\right) g^{(0)}(t-\tau) d \tau
$$

where $\mathcal{I}_{n}(x)$ is the modified Bessel's function of the first kind, and $\alpha=2\langle\varepsilon \sigma\rangle /\langle\varepsilon\rangle$. Performing this calculation using the usual multiple-scale method requires much more algebra.

## Chapter 6

## CONCLUSIONS

### 6.1 Summary of results

We began this paper by asking the question, "How do boundary conditions affect the multiple-scale analysis of hyperbolic conservation laws?"

For systems of hyperbolic conservation laws without rapid spatial fluctuations, we have demonstrated how to construct the solution to an initial-boundary value problem through a judicious choice of temporal and spatial scales. The key finding from Chapter 2 is that in addition to the slow temporal scales traditionally used for an initial-value problem, stretched spatial scales should be used for waves emanating from the boundary. Consistency conditions involving both slow temporal scales and stretched spatial scales should be solved under the assumption that these scales are independent, instead of transforming back to more convenient physical coordinates, in which these extra scales are no longer independent.

For systems of hyperbolic conservation laws with rapid spatial fluctuations, we have highlighted how the difficulty of missing boundary conditions arises whens solving an initial-boundary value problem, and demonstrated how to overcome it. We used two different methods in our analysis: the first the usual multiple-scale method, the second a combination of Laplace transforms and multiple scales (the Laplace-MS method). Using either method, the procedure for recovering boundary information is the same:

1. Use a reduced set of scales to solve the original problem, thereby eliminating the need to solve long-term evolution equations (consistency conditions),
2. generate a temporary solution, ignoring the fact that it has a relatively small region of validity,
3. and extract the necessary boundary information to solve the long-term evolution equations.

We applied this procedure to both linear and nonlinear problems in Chapter 4, using the usual multiple-scale method.

In Chapter 5, we discovered that the missing boundary condition difficulty manifests itself through the Laplace-MS analysis in the form of non-Laplace-invertible terms, and so the Laplace-MS method cannot be used to obtain solutions that account for longterm behaviors like diffusion or dispersion. However, the Laplace-MS method can be used to recover boundary information, and requires much less algebra than the usual multiple-scale method for linear problems with $B\left(x^{*}\right) \neq 0$. The fact that the LaplaceMS method gives the same results as the usual multiple-scale method is to be expected since one can usually interchange derivatives and integrals with respect to $x$ and $t$.

We have seen that the recovered boundary information will contain secular terms in $t$ when the initial conditions are nontrivial. These secular terms limit the region of validity of the recovered boundary information. Fortunately, for linear problems, we can overcome this difficulty with the aid of superposition. By partitioning an initial-boundary value problem into the sum of an initial-value problem and a signaling problem, one can use the recovered boundary information for a signaling problem, which will not contain secular terms, to solve the long-term evolution equations.

The multiple-scale analysis for nonlinear problems (3.3) is no more conceptually difficult, although there are a few procedural complications. First, the linear operator that governs the $x^{*}$-homogenized equations at every order of $\epsilon$ is likely to represent a pair of essentially coupled equations when $B\left(x^{*}\right)$ is nonzero. This coupling causes the general solution to involve Bessel functions, and it obscures the construction of
consistency conditions. Second, initial-boundary value problems for nonlinear PDEs cannot be decomposed into an initial-value problem and a signaling problem. The recovered boundary information for a problem with nontrivial initial conditions is likely to contain secular terms proportional to $t$, which limit the region of validity of the asymptotic expansion. However, since the appearance of diffusion is governed solely by the linear part of the problem, and to this date we have never encountered a physically relevant problem that includes diffusion, we argue that the consistency conditions (long-term evolution equations) for physically relevant nonlinear problems will not include diffusion. Instead, they will form a pair of first-order, quasilinear PDEs. As the order of these consistency conditions matches the original partial differential equation, there is a good chance that no additional boundary conditions are needed to solve the consistency conditions.

### 6.2 Future work

Our current top priority is to build an arsenal of physically relevant problem that are amenable to these solution techniques. We would be especially interested to find a counterexample to our theory in the previous paragraph-a nonlinear problem that exhibits diffusion. The long-term evolution equations for such a problem would form a pair of coupled Burgers' equations, and finding their solution would necessitate the recovery of boundary information.

Since many physical situations are modeled on finite domains, we want to extend this work to problems with more than one boundary. The presence of two boundaries, say $x=0$ and $x=L$, should not pose any greater conceptual difficulties than one.

Another obvious way to extend this work is to generalize our results to problems in more than one spatial dimension, and to systems of more than two conservation laws. Interesting resonance patterns have been observed in systems of three hyper-
bolic conservation laws [17], and these resonances may have significant implications on the boundedness of solutions. Furthermore, we have not considered hyperbolic conservation laws with source terms. Source terms would also have a considerable effect on the boundedness of solutions. (Keep in mind that we are referring to the boundedness of perturbations to steady-state solutions, which implies the stability of these steady-state solutions.)

Throughout this paper, we have only considered problems with periodic fluctuations on the $x^{*}$ scale. We have done so purely for convenience. The same methods and procedures apply to non-periodic fluctuations, as long as the averaging operators presented in Appendix A are still well-defined. In fact, the periodicity of the fluctuations introduces the possibility of resonant interactions between waves traveling through the medium and the medium itself (Section 3.3). In the more distant future, we see that these methods could also be applied to hyperbolic conservation laws with random fluctuations. The averaging operators would have to be modified to describe the cumulative effect of random fluctuations based on statistical measurements of those fluctuations.

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## Appendix A

## AVERAGING OPERATORS

As we will be dealing with functions that vary on the fast spatial scale $x^{*}=x / \epsilon$, we need to introduce the following operators.

$$
\begin{aligned}
& \left\langle a\left(x^{*}\right)\right\rangle \stackrel{\text { def }}{=} \lim _{s \rightarrow \infty} \frac{1}{2 s} \int_{-s}^{s} a(s) d s \quad\left\{a\left(x^{*}\right)\right\} \stackrel{\text { def }}{=} a\left(x^{*}\right)-\left\langle a\left(x^{*}\right)\right\rangle \\
& \llbracket a\left(x^{*}\right) \rrbracket \stackrel{\text { def }}{=} \int_{s}^{x^{*}}\{a(\xi)\} d \xi \text { where } s \text { is chosen such that }\left\langle\left\{a\left(x^{*}\right)\right\}\right\rangle=0
\end{aligned}
$$

The averaging operator $\langle\cdot\rangle$ gives the average value of a function over the entire $x^{*}$ domain. It is a linear operator that produces constants or functions that are independent of $x^{*}$. For a function $b\left(x^{*}\right)$ with period $2 P$, the averaging operator is more conveniently defined as

$$
\left\langle b\left(x^{*}\right)\right\rangle \stackrel{\text { def }}{=} \frac{1}{2 P} \int_{-P}^{P} b(s) d s .
$$

The $\{\cdot\}$ operator generates the fluctuating part of a function: the part of the function that has a zero average. Finally, the $\llbracket \cdot \rrbracket$ operator gives the integral of the fluctuating part of a function, where the constant of integration is chosen such that the average of the integral of the fluctuating part is zero. (Keep in mind that $\left\{a\left(x^{*}\right)\right\}$ is still a function of $x^{*}$, whereas $\left\langle a\left(x^{*}\right)\right\rangle$ is not.) When any of these operators is applied to a matrix, it is understood that the operation is performed element-wise.

We now present some interesting and useful properties of these averaging operators. In the following discussion, it is understood that $a=a\left(x^{*}\right)$ and $b=b\left(x^{*}\right)$ are integrable, although not necessarily periodic or differentiable with respect to $x^{*}$.

Property $1 \frac{d}{d x^{*}} \llbracket a \rrbracket=\{a\}$

This proposition is a direct consequence of the fact that $\llbracket \cdot \rrbracket$ is an integral operator. The function $a\left(x^{*}\right)$ does not have to be differentiable, only integrable.

Property $2 \llbracket \frac{d}{d x^{*}} a\left(x^{*}\right) \rrbracket=\frac{d}{d x^{*}} \llbracket a\left(x^{*}\right) \rrbracket$ for all differentiable $a\left(x^{*}\right)$
To prove this property, we use the definition of the $\llbracket \cdot \rrbracket$ operator:

$$
\llbracket \frac{d}{d x^{*}} a\left(x^{*}\right) \rrbracket=\int_{s}^{x^{*}} a^{\prime}(\xi) d \xi=a\left(x^{*}\right)-a(s),
$$

where $s$ is chosen such that

$$
\left\langle\llbracket \frac{d}{d x^{*}} a\left(x^{*}\right) \rrbracket\right\rangle=\left\langle a\left(x^{*}\right)-a(s)\right\rangle=\left\langle a\left(x^{*}\right)\right\rangle-a(s)=0 .
$$

Substituting $a(s)=\left\langle a\left(x^{*}\right)\right\rangle$,

$$
\llbracket \frac{d}{d x^{*}} a\left(x^{*}\right) \rrbracket=a\left(x^{*}\right)-\left\langle a\left(x^{*}\right)\right\rangle=\left\{a\left(x^{*}\right)\right\}=\frac{d}{d x^{*}} \llbracket a \rrbracket .
$$

In effect, this property allows us to interchange differentiation and integration. It is the only property listed here that requires $a\left(x^{*}\right)$ to be differentiable.

Property $3 \llbracket a \rrbracket \llbracket b \rrbracket=\llbracket \llbracket a \rrbracket\{b\}+\{a\} \llbracket b \rrbracket \rrbracket+\langle\llbracket a \rrbracket \llbracket b \rrbracket\rangle$

Proof.

$$
\begin{aligned}
\qquad \llbracket a \rrbracket \llbracket b \rrbracket & =\{\llbracket a \rrbracket \llbracket b \rrbracket\}+\langle\llbracket a \rrbracket \llbracket b \rrbracket\rangle \\
\qquad \llbracket a \rrbracket \llbracket b \rrbracket\} & =\frac{d}{d x^{*}} \llbracket \llbracket a \rrbracket \llbracket b \rrbracket \rrbracket \quad \text { Use Property } 1 \\
\text { Use Property } 2 & =\llbracket \frac{d}{d x^{*}}(\llbracket a \rrbracket \llbracket b \rrbracket) \rrbracket \\
\text { Use Property } 1 & =\llbracket \llbracket a \rrbracket\{b\}+\{a\} \llbracket b \rrbracket \rrbracket
\end{aligned}
$$

Note: Although this property makes use of Property 2, it does not require $a\left(x^{*}\right)$ or $b\left(x^{*}\right)$ to be differential because if these functions are integrable, then $\llbracket a \rrbracket \llbracket b \rrbracket$ is differentiable.

Property $4\langle\llbracket a \rrbracket a\rangle=0$ for all integrable, scalar functions $a\left(x^{*}\right)$.

Proof.

$$
\begin{aligned}
\langle\llbracket a \rrbracket a\rangle & =\langle\llbracket a \rrbracket(\langle a\rangle+\{a\})\rangle=\langle a\rangle\langle\llbracket a \rrbracket\rangle+\langle\llbracket a \rrbracket\{a\}\rangle=\langle\llbracket a \rrbracket\{a\}\rangle \\
\text { Use Property } 1 & =\left\langle\frac{1}{2} \frac{d}{d x^{*}}\left(\llbracket a \rrbracket^{2}\right)\right\rangle \\
\text { Use Property } 3 & =\frac{1}{2}\left\langle\frac{d}{d x^{*}}\left(2 \llbracket \llbracket a \rrbracket\{a\} \rrbracket+\left\langle\llbracket a \rrbracket^{2}\right\rangle\right)\right\rangle \\
& =\langle\{\llbracket a \rrbracket\{a\}\}\rangle=0
\end{aligned}
$$

Property $5\langle a \llbracket b \rrbracket\rangle=-\langle\llbracket a \rrbracket b\rangle$

We prove this statement by applying Property 4 to the equality

$$
\langle\llbracket a+b \rrbracket(a+b)\rangle=\langle\llbracket a \rrbracket a\rangle+\langle\llbracket a \rrbracket b\rangle+\langle\llbracket b \rrbracket a\rangle+\langle\llbracket b \rrbracket b\rangle .
$$

Since the averaging operator involves an integral, this property is really a manifestation of integration by parts. However, one can define the averaging operator without mention of integrals and the result will still be true. (Simply define $\langle\cdot\rangle$ as a linear operator such that $a\left(x^{*}\right)=\langle a\rangle+\{a\}$ and $\langle\alpha\rangle=\alpha$ for all constants $\alpha$.)

It is very important to remember that Property 4 does not hold for matrices because its proof relies on the commutative property of scalar multiplication. The expression $\langle A \llbracket A \rrbracket\rangle$ appears many times throughout this paper, and this expression is not zero if $A\left(x^{*}\right)$ is a matrix. The most that we can say about $\langle A \llbracket A \rrbracket\rangle$ is that if $A\left(x^{*}\right)$ is a square, integrable matrix, then the trace of $\langle A \llbracket A \rrbracket\rangle$ is zero. (One can prove this quickly using some linear algebra and Properties 4 and 5.)

However, Property 5 does hold for matrices. For any matrices square matrices $A\left(x^{*}\right)$ and $B\left(x^{*}\right)$ of the same size,

$$
\langle A \llbracket B \rrbracket\rangle=-\langle\llbracket A \rrbracket B\rangle .
$$

In particular, if $A=B$, we see that

$$
\langle A \llbracket A \rrbracket\rangle=-\langle\llbracket A \rrbracket A\rangle .
$$

Finally, repeated applications of Property 5 yields this result.
Property $6\langle a \llbracket \llbracket b \rrbracket \rrbracket\rangle=-\langle\llbracket a \rrbracket \llbracket b \rrbracket\rangle=\langle\llbracket \llbracket a \rrbracket \rrbracket b\rangle$ for all integrable $a\left(x^{*}\right)$ and $b\left(x^{*}\right)$

## Appendix B

## NUMERICAL METHODS

In this appendix we give further details about the two numerical methods used throughout this paper: a spectral solver, and a finite volume method.

## B. 1 CLAWPACK

CLAWPACK (Conservation LAWs PACKage), written by Randall J. LeVeque, is a package of Fortran routines that numerically solves hyperbolic systems of conservation laws. (See [24] and [23].) As it is a finite volume method, it can be used to calculate solutions that truly conserve the appropriate quantities. The fundamental unit of CLAWPACK is a user-supplied routine that solves a Riemann problem-an initial-value problem with piece-wise constant data. CLAWPACK handles everything else: timestepping, flux corrections using flux limiters, even adaptive mesh refinement (with AMRCLAW). It is available on the web at

> http : //www. amath. washington.edu/~rjl/clawpack.html.

In this paper, we have employed CLAWPACK to solve the linear system of equations

$$
\begin{equation*}
\mathbf{u}_{t}+A(x / \epsilon) \mathbf{u}_{x}=0 \tag{B.1}
\end{equation*}
$$

where $A(x / \epsilon)$ is a periodic function and $0<\epsilon \ll 1$. The Riemann solver that we have implemented assumes the values of $A\left(x^{*}\right)$ are constant within each finite volume cell, which in turn implies that the wave speeds are different in neighboring cells. This assumption seems to cause the built-in flux limiting routines of CLAWPACK to produce
solutions that grow exponentially in time. Since our analytic results predict that the dispersive effects due to the cumulative fluctuations in $A\left(x^{*}\right)$ only show up for large time, this exponential growth is a serious problem.

As we pointed out in Chapter 3, for some choices of $A\left(x^{*}\right)$, the solution will naturally grow exponentially in time, but we are not referring to these cases. In fact, the linear wave equation with

$$
A_{\text {waves }}\left(x^{*}\right)=\left[\begin{array}{cc}
0 & 1 / \rho\left(x^{*}\right) \\
k\left(x^{*}\right) & 0
\end{array}\right]
$$

has a positive definite, conserved quantity,

$$
\begin{equation*}
E(t)=\int_{-L}^{L}\left[\rho u_{1}^{2}+u_{2}^{2} / k\right] d x \tag{B.2}
\end{equation*}
$$

when solved on a periodic domain $-L<x<L$ (or with zero Dirichlet boundary conditions). In our experiments, we used $E(t)$ as one indicator of the accuracy of the solution calculated by CLAWPACK. We found that none of the built-in flux limiting routines gave a satisfactory answer. We also implemented a transmission-based limiter ${ }^{1}$ devised by Fogarty in [14] and [15], but this scheme seemed to limit fluxes too aggressively, causing the solution to decay exponentially. During a private communication, Dr. LeVeque suggested that the problem might be related to the fact that CLAWPACK does not solve the wave equation in a manner consistent with its conservative form,

$$
\rho\left(x^{*}\right) w_{t t}-\left(k\left(x^{*}\right) w_{x}\right)_{x}=0 .
$$

(Recall that we have used $u_{1}=w_{t}$ and $u_{2}=-k w_{x}$ to convert the second-order PDE to a system of first-order PDEs.)

We found that by far the most satisfactory solution to (B.1) can be obtained by turning off CLAWPACK's built-in flux limiting, and increasing the spatial resolution

[^6]until the solution is largely independent of spatial resolution. For (B.1), increasing the spatial resolution does more than the usual reduction of error; as the width of cells decreases, the properties of the medium become more uniform from cell to cell.

## B. 2 Spectral solver

The other numerical method that we have used in this paper is loosely based on the ideas behind spectral methods. Essentially, we solve the system of ordinary differential equations that results when we assume the solution is a truncated Fourier series.

Consider the system of equations (B.1) with periodic $A(x / \epsilon)$ on the domain $-1<$ $x<1$ with periodic boundary conditions. If the initial conditions $\mathbf{u}(x, 0)$ are periodic, then the solution will also be periodic for all $t>0$. We can therefore write a Fourier series expansion for the solution:

$$
\begin{equation*}
\mathbf{u}(x, t)=\sum_{n=-\infty}^{\infty} \mathbf{v}^{(n)}(t) \cos (n \pi x)+\mathbf{w}^{(n)}(t) \sin (n \pi x) \tag{B.3}
\end{equation*}
$$

Plugging (B.3) into (B.1), we see that the amplitudes $\mathbf{v}^{(n)}(t)$ and $\mathbf{w}^{(n)}(t)$ are governed by an infinite system of ordinary differential equations. The coupling of the amplitudes is governed by $A(x / \epsilon)$.

Equation (B.3) suggests a simple approximation to $\mathbf{u}$ : truncate the Fourier expansion by defining

$$
\begin{equation*}
\tilde{\mathbf{u}}(x, t)=\sum_{n=m}^{M} \mathbf{v}^{(n)}(t) \cos (n \pi x)+\mathbf{w}^{(n)}(t) \sin (n \pi x) \tag{B.4}
\end{equation*}
$$

Now instead of an infinite system of ordinary differential equations, we obtain a system of $2(M-m+1)$ differential equations governing the amplitudes (in much the same way that we obtained a system of equations in Section 3.3). The initial conditions for these ODEs comes from a similar truncated Fourier expansion of the initial conditions to the PDE. The system of ODEs is linear and has constant coefficients, so it can be easily solved.

We implemented this numerical method using Mathematica. Although Mathematica can perform symbolic calculations, in this case the required matrix exponentiation is too computationally intensive for large systems of ODEs. Instead, we have used Mathematica's arbitrary precision arithmetic to calculate solutions for large $t$.

We expect that as more terms are incorporated into the truncated Fourier expansion, the approximation becomes more accurate. But how many Fourier modes should we include? As an illustration, we solved (B.1) with

$$
A_{\text {test }}^{-1}\left(x^{*}\right)=\left[\begin{array}{cc}
1+0.5 \sin \left(\pi x^{*}\right) & 2 \\
2 & 1+0.25 \sin \left(\pi x^{*}\right)
\end{array}\right]
$$

with the initial condition

$$
\mathbf{u}(x, 0)=\left[\begin{array}{c}
\sin (\pi x) \\
0
\end{array}\right]
$$

using $m=-200$ and $M=200$ in the truncated Fourier series (B.4). We used 64digit arithmetic. Figure B. 1 shows the amplitudes of each mode on a logarithmic scale, when $t=500$. There are four dots for each wavenumber because we plot the amplitudes of the cosine and sine for both $\tilde{u}_{1}(x, t)$ and $\tilde{u}_{2}(x, t)$. Also, notice that there are gaps in the graph, indicating that certain wavenumbers are absent from the solution. These gaps are a result of our monochromatic initial condition and $A(x / \epsilon)$-with $\epsilon=0.1$, the initial wave with wavenumber $\pi$ excites only the waves with wavenumbers $\ldots,-19 \pi,-9 \pi, \pi, 11 \pi, 21 \pi, \ldots$

The most important feature of Figure B. 1 is that the amplitudes decay exponentially with increasing wavenumber. If we continue the trend, we see that the magnitude of the first neglected Fourier mode is approximately $10^{-10}$. If we are satisfied with a solution accurate to $10^{-5}$, we only need to set $m=-50$ and $M=50$ in our truncated Fourier series.

We have also used this numerical method to solve the linear wave equation. For a periodic boundary problem, this spectral method is preferable to CLAWPACK because


Figure B.1: Amplitudes of Fourier modes present in a spectral solution to the linear wave equation at $t=500$.
it is able to conserve $E(t)$ perfectly. In general, spectral numerical techniques are typically much more accurate than their finite difference counterparts. However, they are limited to problems with periodic boundary conditions.

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## EDUCATION

Ph. D. in Applied Mathematics; University of Washington, Seattle, Washington, Sept 1996-June 2000

- Washington NASA Space Foundation Grant recipient, 1996-97
- Huckabay Teaching Fellow, 1998-1999
- VIGRE graduate fellowship, 1999-2000
M. S. in Applied Mathematics; Claremont Graduate School, Claremont, California, Jan-June 1996
B. S. in Mathematics, B. A. in Music; Harvey Mudd College, Claremont, California, Sept 1992-June 1996
- Rank in class: 2nd
- Giovanni Fellowship, 1995-96
- Highest Score on the Actuarial Examination, May 1994


## TEACHING EXPERIENCE

University of Washington, Seattle, Washington

- Teaching Assistant for Calculus I (Math 124), Autumn 1996, Winter 1998.
- Lecturer for Applied Linear Algebra and Numerical Analysis (Amath 352), Spring and Summer 1998, Summer 1999.

Amath 352 is a Matlab-based introduction to scientific computing for upper-level undergraduates.

- Lecturer for Communication in the Mathematical Sciences (Amath 500D), a new course that I developed as a Huckabay Teaching Fellow under the supervision of my mentor, Dr. Randall J. LeVeque
- Lecturer for Introduction to Methods in Applied Mathematics III (Amath 403), Spring 1999. This course surveys the analytical techniques used to solve partial differential equations.
- Lecturer for Introduction to Methods in Applied Mathematics I (Amath 401), Fall 1999. We learned vector calculus and complex analysis in this course.
- Participated in a elementary math curricula comparison project (1999-2000), sponsored by the NSF.


## Northshore School District, Seattle, Washington

- Northshore Outreach Program volunteer, Spring and Summer 1998. As a member of an Applied Mathematics Clinic project, I helped to assess K-6 mathematics curricula under review by the Northshore school district.
- Facilitator for a summer institute to help high-school teachers infuse writing into math and science curricula, Summer 2000. Institute director: Gail Robbins.


## RESEARCH EXPERIENCE

Field of specialty: Perturbation theory (multiple-scale analysis), the theory of homogenization, and partial differential equations

College of William and Mary, Williamsburg, Virginia

- Investigation of the factorization of almost-periodic matrix functions arising from Weiner-Hopf integral equations, July-August 1995. Advisor: Dr. Ilya Spitkovsky. Sponsored by the National Science Foundation under the Research Experiences for Undergraduates (REU) program.

University of Washington, Seattle, Washington

- Study of mathematical issues related to the turbulent combustion of gases, Jan-Dec 1997. Advisors: Dr. George Kosaly, Dr. James Riley (Mechanical Engineering department) Sponsored by the Department of Defense (DOD) and Gas Research Institute (GRI).
- Thesis work on boundary-value problems for systems of hyperbolic conservation laws with rapidly fluctuating quantities using the method of multiple scales, Jan 1998-present. Advisor: Dr. Jirair Kevorkian.
- Modeling of a mode-locking optical fiber laser, June 1999-present. Collaboration with Dr. Nathan Kutz and Arnold D. Kim.


## PUBLICATIONS

1. I. Spitkovsky and D. Yong, Almost periodic factorization of certain block triangular matrix functions, Math. Comp., 69 (2000), 1053-1070.

## SELECTED WORK EXPERIENCE

Kumon Math and Reading Centers, Sacramento, California
Head classroom assistant and Computer consultant, Fall 1987-Present.

- Responsible for the purchase, set-up and maintenance of all computer equipment.
- Deployed a database to manage all student, employee records.
- Responsible for the design of monthly newsletter, advertising and teaching materials.
- Additional duties: tutoring, employee training

Environmental Systems Research Institute, Inc., Redlands, California
Team leader for clinic project, Sept 1995-July 1996.

- Team leader for a collaborative project was by the Mathematics Clinic Program at Harvey Mudd College. Advisor: Dr. Lisette de Pillis.
- Project goal: to design, test and code efficient algorithms to solve real-world shortest-path problems for geographical information systems (GIS).

Harvey Mudd College, Claremont, California
Editor for the scholarly journal Interface and newsletter CODEE, Sept 1994-July 1996

- Responsible for the purchase, set-up and maintenance of computer equipment.
- Created a searchable Web index of back-issues.
- Additional duties: staffing, manuscript editing, layout design, typesetting, design of Web pages


## COMPUTER SKILLS

- Languages: C++, Fortran, Assembly, Perl, Scheme (Lisp)
- Mathematics: Matlab, Mathematica, Maple V
- Database: Microsoft Access, SQL
- Publishing: Framemaker, Pagemaker, HTML, InDesign, Illustrator
- Operating Systems: Windows 9x, NT, Unix, Solaris
- Experience with parallel computing (SGI Power Challenge Array)
- Familiar with system administrative duties for Sun workstations


## SPECIAL SKILLS

- American Red Cross certifications: Emergency Responder, CPR
- Knowledge of Chinese and Spanish
- Avid interpreter of Brahms' piano works


## VOLUNTEER WORK

- Catholic Community Services
- Volunteer tutoring at True Vine Church of God in Christ
- United Gospel Mission
- InterVarsity Christian Fellowship
- Treasurer for Windermere North Condominiums


[^0]:    ${ }^{1}$ We wish to remind the reader that the remainder term in every asymptotic expansion is more an indication of the rate of convergence of the truncated expansion, than the accuracy of the truncated expansion. In other words, the fact that the $\mathcal{O}\left(\epsilon^{3}\right)$ terms are omitted in (2.5) means that the truncated series up to the $\epsilon^{2}$ contribution will converge like $\epsilon^{3}$ as $\epsilon \rightarrow 0$. In the rest of this paper, we will omit the reminder " $\epsilon \rightarrow 0$," which the reader should implicitly assume anytime the symbol $\mathcal{O}\left(\epsilon^{n}\right)$ appears in an asymptotic expansion.

[^1]:    ${ }^{2}$ In the context of celestial mechanics, secular terms account for long-term deviations from the most prominent periodic features of the object being studied. For example, the presence of other planets in our solar system causes the earth's orbit around the sun to precess at a very slow rate [31].

[^2]:    ${ }^{3}$ If there were source terms in (2.17), they would manifest themselves here as an additional matrix multiplying $\mathbf{u}^{(1)}$, essentially coupling this pair of hyperbolic equations. While some progress could be made, the general solution would be very complicated as it would involve Bessel functions. See Section 3.7.2 of [18].

[^3]:    ${ }^{1}$ We did not include corrections to the initial conditions (4.14) discussed in Section 3.4 for the numerical solution. These corrections would only improve the correspondence between the analytic and numeric solutions.

[^4]:    ${ }^{1}$ In Section 4.2 we explained why we don't lose any generality by specifying $u_{1}$ instead of a linear combination of $u_{1}$ and $u_{2}$ at $x=0$.

[^5]:    ${ }^{2}$ The eigenvalue $\lambda_{2}$ is associated with the negative sign in (5.16). Also, when $|s|$ is large, the eigenvalues of $A^{-1}$ will dominate and we have assumed that one eigenvalue of $A^{-1}$ is positive and the other is negative.

[^6]:    ${ }^{1}$ The transmission-based limiter works well in situations in which the properties of the medium are vastly different from one cell to the next. It works by breaking up each wave into its transmitted and reflected parts, and comparing the transmitted parts of waves in each cell with their neighbors.

