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HW # 9
Math 61 Sec 2
Due: 5/31/2004

Section 7.3 # 1, 2, 5, 10 (Stoke's Theorem)
Section 7.3 # 9, 11, 12, 14 (Gauss's Divergence Theorem)

1-2) Verify Stoke's theorem for the given surface and vector field

1) S is defined by $x^2+y^2+5z=1$, $z \geq 0$, oriented by upward normal;

$$\vec{F} = xz\,\hat{i} + yz\,\hat{j} + (x^2+y^2)\,\hat{k}$$

S: $x^2+y^2+5z=1$
If $z \geq 0$, it is clear that ∂S is defined over the x-y plane when $z=0$
So let $z=0$
$x^2+y^2+5\cdot 0=1 \Rightarrow x^2+y^2=1$
So ∂S consists of the circle $C=\{(x,y,z) \mid x^2+y^2=1,\ z=0\}$
By Stoke's Theorem, then

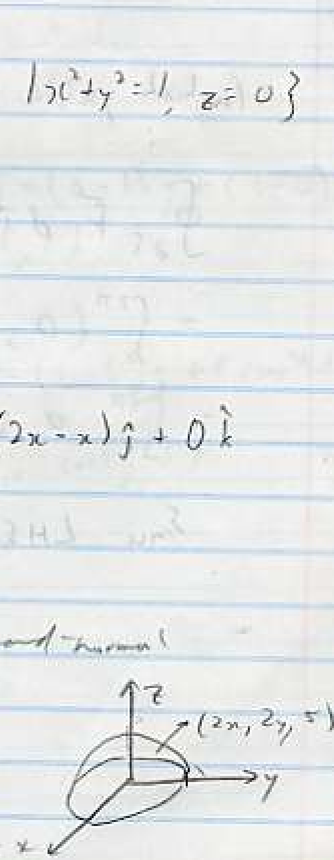
$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

LHS: $\vec{F} = xz\,\hat{i} + yz\,\hat{j} + (x^2+y^2)\,\hat{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & x^2+y^2 \end{vmatrix} = (2y-y)\,\hat{i} - (2x-x)\,\hat{j} + 0\,\hat{k} = (y, -x, 0)$$

Find the gradient of x^2+y^2+5z to find its upward normal

$\nabla S = (2x, 2y, 5)$, in which oriented upward



By using formula (5) of §7.2, we have, where $D = \{(x,y) \mid x^2+y^2 \le 1\}$

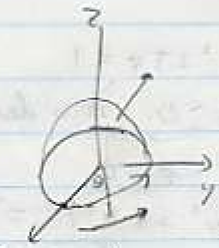
$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_D (y, -x, 0) \cdot (2x, 2y, 5)\, dx\, dy$$

$$= \iint_D (2xy - 2xy + 0)\, dx\, dy$$

$$= \iint_D 0\, dx\, dy = \boxed{0}$$

RHS: We may parametrize ∂S as

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \qquad 0 \le \theta \le 2\pi \\ z = 0 \end{cases}$$



(which yields the orientation desired for ∂S)

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(x(t)) \cdot x'(t)\, dt$$

$$= \int_0^{2\pi} (0, 0, \cos^2\theta + \sin^2\theta) \cdot (-\sin\theta, \cos\theta, 0)\, dt$$

$$= \int_0^{2\pi} 0\, dt = \boxed{0}$$

Since LHS = RHS, Stoke's theorem thus holds.

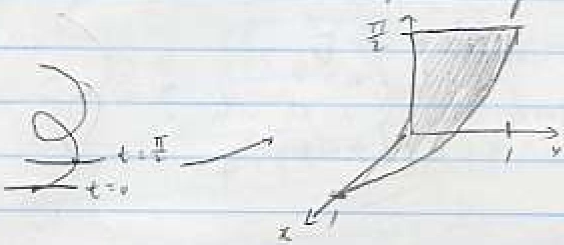
2) S is parametrized by $\vec{X}(s,t) = (s\cos t,\ s\sin t,\ t)$,

$0 \le s \le 1 \quad 0 \le t \le \pi/2$

$\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$

$$\begin{cases} x = s\cos t \\ y = s\sin t & 0 \le s \le 1 \\ z = t & 0 \le t \le \frac{\pi}{2} \end{cases}$$

S is a helix which stops at a quarter circle



By Stoke's Theorem

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

LHS

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = (1-0)\hat{i} - (0-1)\hat{j} + (1-0)\hat{k} = (1,1,1)$$

$$X_s \times X_t \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos t & \sin t & 0 \\ -s\sin t & s\cos t & 1 \end{vmatrix} = (\sin t)\hat{i} - (\cos t)\hat{j} + (s\cos^2 t + s\sin^2 t)\hat{k} = (\sin t, -\cos t, s)$$

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{F} \cdot (X_s \times X_t)\, ds\, dt$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 (1,1,1) \cdot (\sin t, -\cos t, s)\, ds\, dt$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 \sin t - \cos t + s \quad ds\, dt$$

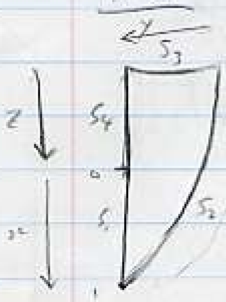
$$= \int_0^{\frac{\pi}{2}} \left(s\sin t - s\cos t + \frac{s^2}{2} \right)\Big|_0^1 dt$$

$$= \int_0^{\frac{\pi}{2}} \sin t - \cos t + \frac{1}{2}\, dt = -\cos t - \sin t + \frac{1}{2} t \Big|_0^{\frac{\pi}{2}}$$

$$= \left(-\cos\frac{\pi}{2} - \sin\frac{\pi}{2} + \frac{1}{2}\cdot\frac{\pi}{2} \right) - \left(-\cos 0 - \sin 0 + \frac{1}{2}\cdot 0 \right)$$

$$= \left(-1 + \frac{\pi}{4} \right) - (-1) = \boxed{\frac{\pi}{4}}$$

RHS



$$\oint_{\partial S} \vec{F}\cdot d\vec{s} = \int_{S_1} \vec{F}\cdot d\vec{s} + \int_{S_2} \vec{F}\cdot d\vec{s} + \int_{S_3} \vec{F}\cdot d\vec{s} + \int_{S_4} \vec{F}\cdot d\vec{s}$$

Parameterize S_1, S_2, S_3, S_4

$$S_1 \begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases} \quad t\ 0 \to 1 \qquad S_2 \begin{cases} x = \cos t \\ y = \sin t \\ z = t \end{cases} \quad t\ 0 \to \frac{\pi}{2}$$

$$S_3 \begin{cases} x = 0 \\ y = t \\ z = \frac{\pi}{2} \end{cases} \quad t\ 1 \to 0 \qquad S_4 \begin{cases} x = 0 \\ y = 0 \\ z = t \end{cases} \quad t\ \frac{\pi}{2} \to 0$$

$$\int_{S_1} \vec{F}\cdot d\vec{s} = \int_{S_1} \vec{F}\cdot x'(t)\,dt = \int_{S_1} (z, x, y)\cdot(1,0,0)\,dt$$

$$\vec{x}(t) = (t, 0, 0)$$

$$= \int_{S_1} z\,dt = \int_{S_1} 0\,dt = 0$$

$$\int_{S_2} \vec{F}\cdot d\vec{s} = \int_{S_2} (z, x, y)\cdot(-\sin t, \cos t, 1)\,dt = \int_{S_2} (-z\sin t + x\cos t + y)\,dt$$

$$\vec{x}(t) = (\cos t, \sin t, t)$$

$$= \int_{S_2} (-t\sin t + \cos^2 t + \sin t)\,dt$$

$$= \int_0^{\frac{\pi}{2}} -t\sin t\,dt + \int_0^{\frac{\pi}{2}} \cos^2 t\,dt + \int_0^{\frac{\pi}{2}} \sin t\,dt \overset{\text{TI-89}}{=} \frac{\pi}{4}$$

$$\int_{S_3} \vec{F}\cdot d\vec{s} = \int_{S_3} (z, x, y)\cdot(0, 1, 0)\,dt = \int_{S_3} x\,dt = \int_{S_3} 0\,dt = 0$$

$$\vec{x}(t) = (0, t, \tfrac{\pi}{2})$$

$$\int_{S_4} \vec{F}\cdot d\vec{s} = \int_{S_4} (z, x, y)\cdot(0, 0, 1)\,dt = \int_{S_4} y\,dt = \int_{S_4} 0\,dt = 0$$

$$\vec{x}(t) = (0, 0, t)$$

$$\oint_{\partial S} \vec{F}\,d\vec{s} = 0 + \frac{\pi}{4} + 0 + 0 = \boxed{\frac{\pi}{4}}$$

$\therefore$ LHS = RHS, verifying Stoke's theorem.

c) Let S be the "silo surface," that is, S is the union of two smooth surfaces S_1 and S_2

S_1 : $x^2 + y^2 = 9 \quad 0 \le z \le 8$

S_2 : $x^2 + y^2 + (z-8)^2 = 9 \quad z \ge 8$

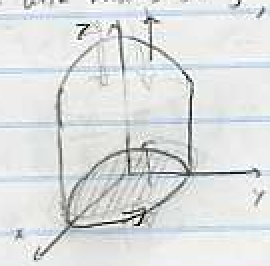
Find $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$, where

$$\vec{F} = (x^3 + xz + yz^2)\,\hat{\imath} + (xyz^3 + y^7)\,\hat{\jmath} + x^2 z^5\,\hat{k}$$

$\iint_S \nabla \times \vec{F} \cdot d\vec{S}$

S_1 : It is a cylinder with a radius of 3 with length from $z = 0 \to 8$

S_2 : It is an upper-half of a sphere with radius of 3, centered at $(0, 0, 8)$



By Stoke's theorem, $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$ where ∂S consists of the circle (like a projection of S onto the xy-plane) $C = \{(x, y, z) \mid x^2 + y^2 = 9, z = 0\}$.

$\Rightarrow \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{\partial S'} \vec{F} \cdot d\vec{s} = \iint_{S'} \nabla \times \vec{F} \cdot d\vec{S}$

where we let S' be just the circle C.

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_{S'} \nabla \times \vec{F} \cdot d\vec{S}$$

$$= \iint_{S'} (\nabla \times \vec{F} \cdot \hat{n})\, dS$$

We orient S' by $\hat{n} = \underline{(0, 0, 1)}$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3+xz+yz^2 & xyz^3+y^7 & x^2z^5 \end{vmatrix} = (0 - 3xyz^2)\,\hat{\imath} - (2xz^5 - (x+2yz))\,\hat{\jmath} + (yz^3 - z^2)\,\hat{k}$$

$$= (-3xyz^2,\ x+2yz-2xz^5,\ yz^3$$

$$\iint_{S'} (\nabla \times \vec{F} \cdot \hat{n})\, dS = \iint_{S'} (yz^3 - z^2)\, dS$$

and since C is $x^2+y^2=9,\ z=0$

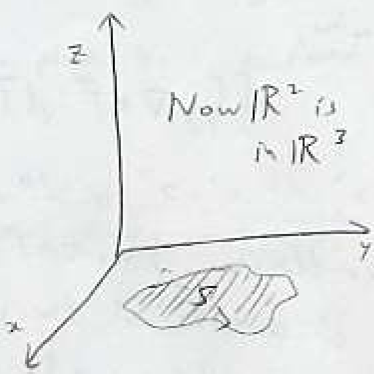
$$\Rightarrow \iint_{S'} (y(0)^3 - 0^2)\, dS = \boxed{0}$$

10.) Verify that Stoke's theorem implies Green's theorem

Assume $\vec{F}$ is independent of z and its k-component is identically zero.

$\Rightarrow \vec{F}(x,y,z) = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath} + 0\hat{k}$

Stoke's theorem states that



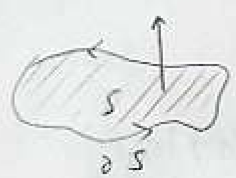
$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

where ∂S is the boundary of the surface S.

$\Rightarrow$ LHS:

$\iint_S \nabla \times \vec{F} \cdot d\vec{S} \qquad \vec{F} = M(x,y)\hat{\imath} + N(x,y)\hat{\jmath}$, so

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M(x,y) & N(x,y) & 0 \end{vmatrix} = \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$



For this $\mathbb{R}^2$ region S
to be oriented properly,
the normal has to be pointed upwards,
so $\hat{n} = (0,0,1)$.

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, dS = \iint_S \left(0, 0, \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) \cdot (0,0,1)\, dS$$

$$= \iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\, dy$$

$\Rightarrow$ RHS.

$$\oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{\partial S} (M, N, 0) \cdot (dx, dy, dz) = \oint_{\partial S} M\,dx + N\,dy$$

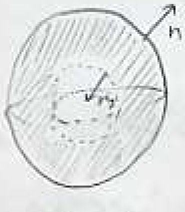
$\Rightarrow$ RHS = LHS

$\oint_{\partial S} M\,dx + N\,dy = \iint_S \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\, dx\, dy$, which verifies that Green's Theorem is a special case of Stoke's Theorem, where $\vec{F}$ is <u>planar field</u> (independent of z).

9) Verify Gauss's theorem for the given three-dimensional region D and vector field $\vec{F}$.

$$\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} \qquad D = \{(x,y,z) \mid a^2 \le x^2+y^2+z^2 \le b^2\}$$

D is the region between sphere with radius a and sphere with radius b.



$\Rightarrow a<b$

Then, by Gauss's theorem

$$\oiint_{\partial D} \vec{F}\cdot d\vec{S} = \iiint_D \nabla\cdot\vec{F}\,dV$$

LHS

$$\oiint_{\partial D} \vec{F}\cdot d\vec{S} = \oiint_{\partial D_1} \vec{F}\cdot d\vec{S} + \oiint_{\partial D_2} \vec{F}\cdot d\vec{S}$$

where D_1 is the boundary of the outer sphere and ∂D_2 is the boundary of the inner sphere

The unit normal vector has to be directed such that it is pointing away from the region D. Thus n_1 has to be pointing outwards and n_2 has to be inwards.

$\partial D_1 \Rightarrow x^2+y^2+z^2 = b^2$

$\partial D_2 \Rightarrow x^2+y^2+z^2 = a^2$

$$\oiint_{\partial D_1} \vec{F}\cdot d\vec{S} = \oiint_{\partial D_1} \left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}\right) d\vec{S} = \oiint_{\partial D_1} \left(\frac{x}{b}, \frac{y}{b}, \frac{z}{b}\right) d\vec{S}$$

S_1 $\begin{cases} x = b\cos\theta\sin\varphi \\ y = b\sin\theta\sin\varphi \\ z = b\cos\varphi \end{cases}$ $\quad \varphi: 0 \to \pi$, $\theta: 0 \to 2\pi$ $\quad X = (b\cos\theta\sin\varphi, b\sin\theta\sin\varphi, b\cos\varphi)$

$$X_\theta \times X_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -b\sin\theta\sin\varphi & b\cos\theta\sin\varphi & 0 \\ b\cos\theta\cos\varphi & b\sin\theta\cos\varphi & -b\sin\varphi \end{vmatrix} = (-b^2\sin^2\varphi\cos\theta)\hat{i} - (b^2\sin^2\varphi\sin\theta)\hat{j} + (-b^2\sin^2\theta\sin\varphi\cos\varphi - b^2\cos^2\theta\sin\varphi\cos\varphi)\hat{k}$$

$$= (-b^2\sin^2\varphi\cos\theta, -b^2\sin^2\varphi\sin\theta, -b^2\sin\varphi\cos\varphi)$$

But since n_1 is pointed outwards, $n_1 = (b^2\sin^2\varphi\cos\theta, b^2\sin^2\varphi\sin\theta, b^2\sin\varphi\cos\varphi)$

$$\Rightarrow \oiint_{\partial D_1} \left(\frac{b\cos\theta\sin\varphi}{b}, \frac{b\sin\theta\sin\varphi}{b}, \frac{b\cos\varphi}{b}\right)\cdot(b^2\sin^2\varphi\cos\theta, b^2\sin^2\varphi\sin\theta, b^2\sin\varphi\cos\varphi)\,d\varphi\,d\theta$$

$$= \oiint_{\partial D_1} b^2\sin^3\varphi\cos^2\theta + b^2\sin^3\varphi\sin^2\theta + b^2\sin\varphi\cos^2\varphi\,d\varphi\,d\theta$$

$= \oiint_{\partial D} b^2 \sin\varphi (\sin^2\varphi \cos^2\theta + \sin^2\varphi \sin^2\theta + \cos^2\varphi)\, d\varphi\, d\theta$

$= \oiint_{\partial D} b^2 \sin\varphi \, d\varphi \, d\theta = \int_0^{2\pi}\int_0^{\pi} b^2 \sin\varphi \, d\varphi\, d\theta$

$= b^2 \int_0^{2\pi} -\cos\varphi \Big|_0^{\pi} d\theta = b^2 \int_0^{2\pi} 2\, d\theta = \underline{4\pi b^2}$

$S_2 \begin{cases} x = a\cos\theta\sin\varphi & \varphi: 0 \to \pi \\ y = a\sin\theta\sin\varphi & \theta: 0 \to 2\pi \\ z = a\cos\varphi & \end{cases}$

Since n_2 is pointed inwards, $n_2 = (-a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, -a^2\sin\varphi\cos\varphi)$

$\oiint_{\partial D_2} \vec{F} \cdot d\vec{S} = \oiint_{D_2} \left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) \cdot (-a^2\sin^2\varphi\cos\theta, -a^2\sin^2\varphi\sin\theta, -a^2\sin\varphi\cos\varphi)\, d\varphi\, d\theta$

$= -\oiint_{\partial D} a^2 \sin\varphi \, d\varphi\, d\theta = -\int_0^{2\pi}\int_0^{\pi} a^2 \sin\varphi \, d\varphi\, d\theta = \underline{-4\pi a^2}$

$\oiint_{\partial D} \vec{F}\cdot d\vec{S} = \oiint_{\partial D_1} \vec{F}\cdot d\vec{S} + \oiint_{\partial D_2} \vec{F}\cdot d\vec{S} = 4\pi b^2 - 4\pi a^2 = \underline{4\pi(b^2 - a^2)}$

9 cont.)

RHS: $\iiint_D \nabla\cdot\vec{F}\,dV$

$$\nabla\cdot\vec{F} = \left(\frac{\partial\, x/\sqrt{x^2+y^2+z^2}}{\partial x} + \frac{\partial\, y/\sqrt{x^2+y^2+z^2}}{\partial y} + \frac{\partial\, z/\sqrt{x^2+y^2+z^2}}{\partial z} \right)$$

$$= \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}} + \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{2x^2+2y^2+2z^2}{(x^2+y^2+z^2)^{3/2}} = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{3/2}} = \frac{2}{(x^2+y^2+z^2)^{1/2}}$$

Change to spherical coordinates

$$\begin{cases} x = \rho\cos\theta\sin\varphi & \rho: a\to b \\ y = \rho\sin\theta\sin\varphi & \theta: 0\to 2\pi \\ z = \rho\cos\varphi & \varphi: 0\to\pi \end{cases} \Rightarrow x^2+y^2+z^2 = \rho^2$$

$$\iiint_D \nabla\cdot\vec{F}\,dV = \iiint_D \nabla\cdot\vec{F}\,dx\,dy\,dz = \iiint_{D^*} \nabla\cdot\vec{F} \left| \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} \right| d\rho\,d\varphi\,d\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \det\begin{bmatrix} x_\rho & x_\varphi & x_\theta \\ y_\rho & y_\varphi & y_\theta \\ z_\rho & z_\varphi & z_\theta \end{bmatrix} = \det\begin{bmatrix} \sin\varphi\cos\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta \\ \sin\varphi\sin\theta & \rho\cos\varphi\sin\theta & \rho\sin\varphi\cos\theta \\ \cos\varphi & -\rho\sin\varphi & 0 \end{bmatrix}$$

$$= \cos\varphi\,(\rho^2\cos^2\theta\sin\varphi\cos\varphi + \rho^2\sin^2\theta\sin\varphi\cos\varphi)$$
$$+ \rho\sin\varphi\,(\rho\cos^2\theta\sin^2\varphi + \rho\sin^2\theta\sin^2\varphi)$$

$$= \rho^2\cos\varphi(\sin\varphi\cos\varphi) + \rho^2\sin^3\varphi = \rho^2\sin\varphi(\cos^2\varphi + \sin^2\varphi) = \rho^2\sin\varphi$$

and $\sin\varphi$ is always nonnegative from $0\to\pi$, so

$$\Rightarrow \iiint_D \nabla\cdot\vec{F}\,dV = \int_0^{2\pi}\int_0^{\pi}\int_a^b \frac{2}{\rho}\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \qquad \left| \frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} \right| = \rho^2\sin\varphi$$

$$= \int_0^{2\pi}\int_0^{\pi} \cancel{2}\frac{\rho^2}{\cancel{2}}\sin\varphi \Big|_a^b d\varphi\,d\theta = \int_0^{2\pi}\int_0^{\pi}(b^2-a^2)\sin\varphi\,d\varphi\,d\theta = (b^2-a^2)\int_0^{2\pi} -\cos\varphi\Big|_0^{\pi}\,d\theta$$

$$= (b^2-a^2)\int_0^{2\pi} 2\,d\theta = \boxed{4\pi(b^2-a^2)}$$

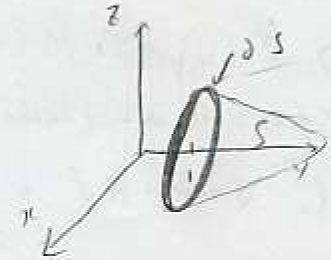
$\therefore$ LHS = RHS, verifying Gauss's theorem

11.) Let S be defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with rightward pointing normal. Let

$$\vec{F} = (2xyz + 5z)\,\hat{i} + e^x \cos yz\,\hat{j} + x^2 y\,\hat{k}$$

Determine

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S}$$



$y = 10 - x^2 - z^2 \Rightarrow x^2 + z^2 + y = 10$, so when $y = 1$, $x^2 + z^2 = 9$

By Stoke's theorem, $\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$, where

∂S consists of the circle

$$C = \{(x, y, z) \mid x^2 + z^2 = 9,\ y = 1\}$$

$$\Rightarrow \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \oint_{\partial S'} \vec{F} \cdot d\vec{s} = \iint_{S'} \nabla \times \vec{F} \cdot d\vec{S}$$

where we let S' be the circle C

$$\Rightarrow \iint_S \nabla \times \vec{F} \cdot d\vec{S} = \iint_{S'} \nabla \times \vec{F} \cdot d\vec{S}$$

$$= \iint_{S'} (\nabla \times \vec{F} \cdot \hat{n})\, dS$$

We orient S' by $\hat{n} = (0, 1, 0)$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz + 5z & e^x \cos yz & x^2 y \end{vmatrix} = (x^2 + y e^x \sin yz)\,\hat{i} - (2xy - (2xy + 5))\,\hat{j} + (e^x \cos yz - 2xz)\,\hat{k}$$

$$= (x^2 + y e^x \sin yz,\ 5,\ e^x \cos yz - 2xz)$$

$$\Rightarrow \iint_{S'} (\nabla \times \vec{F})\, dS = \iint_{S'} (x^2 + y e^x \sin yz, 5, e^x \cos yz - 2xz) \cdot (0, 1, 0)\, dS$$

$$= \iint_{S'} 5\, dS = 5 \cdot \underbrace{\text{area of } S'}_{\text{area of circle with radius } 3} = 5(9\pi) = \boxed{45\pi}$$

12) Let S be defined by $z = e^{1-x^2-y^2}$, $z \geq 1$, oriented by upward normal. and let $\vec{F} = x\,\hat{i} + y\,\hat{j} + (2-2z)\,\hat{k}$. Use Gauss's theorem to calculate

$$\iint_S \vec{F}\cdot d\vec{S}.$$

Consider the piecewise smooth, closed surface created by taking the union of S and S', where S' is the portion of the plane $z=1$ enclosed by ∂S

$1 = e^{1-(x^2+y^2)} \implies$ the disk $x^2+y^2 \leq 1$, $z=1$.

S is oriented by upward, or outerward normal, so S' must also be oriented outward, or downward normal $\hat{n} = (0,0,-1)$

Then, by Gauss's theorem,

$$\iint_S \vec{F}\cdot d\vec{S} + \iint_{S'} \vec{F}\cdot d\vec{S} = \oiint_{\partial D} \vec{F}\cdot d\vec{S} = \iiint_D \nabla\cdot\vec{F}\,dV$$

$$\nabla\cdot\vec{F} = (1+1-2) = 0$$

$$\implies \iint_S \vec{F}\cdot d\vec{S} + \iint_{S'} \vec{F}\cdot d\vec{S} = 0,$$

so $\iint_S \vec{F}\cdot d\vec{S} = -\iint_{S'} \vec{F}\cdot(0,0,-1)\,dS = \iint_R (2-2z)\,dx\,dy$

where R is the unit disk $x^2+y^2 \leq 1$ in the plane $z=1$.

$$\implies \iint_R (2-2z)\,dx\,dy = \iint_R (2-2\cdot 1)\,dx\,dy = \iint_R 0\,dx\,dy = \boxed{0}$$

14) Use Gauss's theorem to evaluate

$$\iint_S \vec{F} \cdot d\vec{S},$$

where $\vec{F} = ze^{x^2}\hat{i} + 3y\hat{j} + (2 - yz^7)\hat{k}$ and S is the union of the five "upper" faces of the unit cube $[0,1] \times [0,1] \times [0,1]$.

That is, the $z=0$ face is not part of S.

Take the union of S and S', where S' is the portion of the plane $z=0$ enclosed by ∂S

Then by Gauss's theorem

$$\iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S} = \oiint_{\partial D} \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F}\, dv$$

$$\nabla \cdot \vec{F} = 2xze^{x^2} + 3 - 7yz^6$$

Notice that D is the unit cube $[0,1] \times [0,1] \times [0,1]$

So $x: 0 \to 1$

$y: 0 \to 1$

$z: 0 \to 1$

$$\iiint_D \nabla \cdot \vec{F}\, dV = \int_0^1 \int_0^1 \int_0^1 2xze^{x^2} + 3 - 7yz^6\, dx\, dy\, dz \; *$$

$$= \int_0^1 \int_0^1 \left(ze^{x^2}\Big|_0^1 + 3x\Big|_0^1 - 7yz^6x\Big|_0^1 \right) dy\, dz$$

$$= \int_0^1 \int_0^1 ze - z + 3 - 7yz^6\, dy\, dz$$

$$= \int_0^1 ze - z + 3 - \left(\frac{7y^2z^6}{2}\right)\Big|_0^1 dz$$

$$= \int_0^1 ze - z + 3 - \frac{7z^6}{2}\, dz$$

$$= \frac{z^2e}{2}\Big|_0^1 - \frac{z^2}{2}\Big|_0^1 + 3 - \frac{7z^7}{14}\Big|_0^1$$

$$= \frac{e}{2} - \frac{1}{2} + 3 - \frac{1}{2} = \frac{e}{2} + 2$$

$*$ $\int 2xze^{x^2}\,dx$

Let $u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$\int 2xze^{u}\frac{du}{2x} = ze^{x^2}$

$$\iint_{S'} \vec{F} \cdot d\vec{S} = \int_0^1\int_0^1 (ze^{x^2}, 3y, 2 - yz^7) \cdot (0,0,-1)\, ds\, dt$$

$$= \int_0^1 \int_0^1 yz^7 - 2\, ds\, dt$$

$$= \int_0^1 \int_0^1 t(0)^7 - 2\, ds\, dt = \int_0^1\int_0^1 -2\, ds\, dt = -2$$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{S} = \iiint_D \nabla \cdot \vec{F}\, dv - \iint_{S'} \vec{F} \cdot d\vec{S}$$

$$= \frac{e}{2} + 2 - (-2) = \boxed{\frac{e}{2} + 4}$$

Parametrize S'

$\begin{cases} x = s & s: 0 \to 1 \\ y = t & t: 0 \to 1 \\ z = 0 \end{cases}$

$$X_s \times X_t = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = (0,0,1)$$

but the normal vector must be oriented away from the region so it must be $(0,0,-1)$