Key to Quiz I (Summer Math 61)

#1 sln

Find critical pts:

$$\nabla f = 0 \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 18x^2 - 6y = 0 \\ \frac{\partial f}{\partial y} = -6x - 2y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - y = 0 & \cdots ① \\ 3x + y = 0 & \cdots ② \end{cases}$$

$$① + ② \Rightarrow 3x^2 + 3x = 0 \Rightarrow x^2 + x = 0 \Rightarrow x(x+1) = 0$$

$$\Rightarrow x = 0 \text{ or } x = -1$$

If $x = 0$, $② \Rightarrow y = 0$
If $x = -1$, $② \Rightarrow y = 3$
$\Rightarrow$ Critical pts $(0,0)$, $(-1,3)$.

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$$\text{Now } Hf = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 36x & -6 \\ -6 & -2 \end{bmatrix}$$

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$$Hf(0,0) = \begin{bmatrix} 0 & -6 \\ -6 & -2 \end{bmatrix}$$

$\Rightarrow \det Hf(0,0) = -36 < 0$

$\Rightarrow (0,0)$ is a saddle pt.

3

$$Hf(-1,3) = \begin{bmatrix} -36 & -6 \\ -6 & -2 \end{bmatrix}$$

$\det Hf(-1,3) = 36 \times 2 - 36 = 36 > 0$

$f_{xx}(-1,3) = -36 < 0$

$\Rightarrow (-1,3)$ local max pt.

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#2 Soln

Want to max/min $f(x,y,z) = x^2 + y^2$ under the constraint $g(x,y) = 5x^2 + 6xy + 5y^2 - 4 = 0$

Note The set $\{(x,y) \mid 5x^2 + 6xy + 5y^2 = 4\}$ is compact since it is an ellipse. To see this, you can complete the square as following: $x^2 + \frac{6}{5}xy + y^2 = \frac{5}{4}$

$$\Rightarrow x^2 + \frac{6}{5}xy + \left(\frac{3}{5}\right)^2 y^2 - \left(\frac{3}{5}\right)^2 y^2 + y^2 = \frac{5}{4}$$

$$\Rightarrow \left(x + \frac{3}{5}y\right)^2 + \left(\frac{4}{5}\right)^2 y^2 = \frac{5}{4}.$$

So there ∃ global max ^{m} and global min

Step 1 $\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$ 5

$$\Rightarrow \begin{cases} \begin{pmatrix} 2x \\ 2y \end{pmatrix} = \lambda \begin{pmatrix} 10x + 6y \\ 6x + 10y \end{pmatrix} \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} 2x = 2\lambda(5x + 3y) \\ 2y = 2\lambda(3x + 5y) \\ g(x,y) = 0 \end{cases}$$ .5

$$\Rightarrow \begin{cases} x = \lambda(5x + 3y) \\ y = \lambda(3x + 5y) \\ g(x,y) = 0 \end{cases} \Rightarrow \begin{cases} (5\lambda - 1)x + 3\lambda y = 0 \\ 3\lambda x + (5\lambda - 1)y = 0 \end{cases} \overset{(*)}{\Rightarrow} \begin{bmatrix} 5\lambda - 1 & 3\lambda \\ 3\lambda & 5\lambda - 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Downarrow$

Note $\begin{cases} x = 0 \\ y = 0 \end{cases}$ is not a solution since $g(0,0) \neq 0$

Thus the homogeneous system has nonzero solution. This happens iff

$$\det \begin{bmatrix} 5\lambda - 1 & 3\lambda \\ 3\lambda & 5\lambda - 1 \end{bmatrix} = 0 \Rightarrow (5\lambda - 1)^2 - 9\lambda^2 = 0$$

$$\Rightarrow 16\lambda^2 - 10\lambda + 1 = 0 \Rightarrow \lambda_{1,2} = \frac{10 \pm \sqrt{100 - 4 \times 16}}{2 \times 16}$$

$$\Rightarrow \lambda_{1,2} = \frac{10 \pm 6}{2 \times 16}$$

$$\Rightarrow \lambda_1 = \frac{1}{2}, \quad \lambda_2 = \frac{1}{8}$$

For $\lambda_1 = \frac{1}{2}$

$(*) \Rightarrow \frac{3}{2}x + \frac{3}{2}y = 0$

$\Rightarrow x + y = 0$

$\Rightarrow y = -x$

For $\lambda_2 = \frac{1}{8}$

$(*) \Rightarrow -\frac{3}{8}x + \frac{3}{8}y = 0$

$\Rightarrow -x + y = 0$

$\Rightarrow y = x$

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when $y = x$,

$5x^2 + 6xy + 5y^2 = 4 \Rightarrow 5x^2 + 6x^2 + 5x^2 = 4$

2 $\Rightarrow 16x^2 = 4 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$

$\Rightarrow (1/2, 1/2), (-1/2, -1/2)$ are critical pts.

when $y = -x$

2 $5x^2 - 6x^2 + 5x^2 = 4 \Rightarrow 4x^2 = 4$

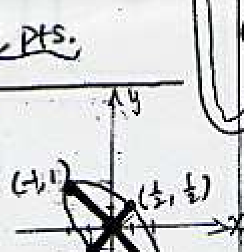
$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\Rightarrow (1,-1), (-1,1)$ are critical pts.

$f(1/2, 1/2) = f(-1/2, -1/2) = \frac{1}{2}$

5 $f(1,-1) = f(-1,1) = 2$

$\Rightarrow (1/2, 1/2), (-1/2, -1/2)$ global min



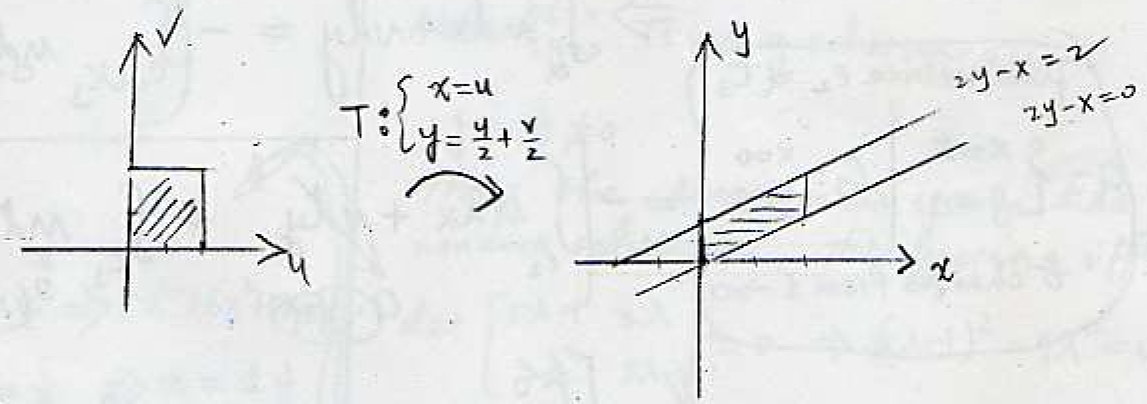
#3 sln

Let $\begin{cases} u = x \\ v = 2y - x \end{cases} \Rightarrow \begin{cases} x = u \\ y = \frac{u}{2} + \frac{v}{2} \end{cases} \Rightarrow \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$

the original region that the integral taking place is

$\begin{cases} \text{from } y = \frac{x}{2} \text{ to } y = \frac{x}{2} + 1 \\ \text{from } x = 0 \text{ to } x = 2 \end{cases}$ i.e. $\begin{cases} \text{from } 2y - x = 0 \text{ to } 2y - x = 2 \\ \text{from } x = 0 \text{ to } x = 2 \end{cases}$

Thus $\begin{cases} 0 \le u \le 2 \\ 0 \le v \le 2 \end{cases}$



Thus $I = \int_{v=0}^{2} \int_{u=0}^{2} u^5 v\, e^{v^2} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$

$= \int_{u=0}^{2} u^5\, du \int_{v=0}^{2} v e^{v^2} \frac{1}{2}\, dv$

$= \left(\frac{u^6}{6} \Big|_{u=0}^{2} \right) \left(\frac{1}{4} e^{v^2} \Big|_{v=0}^{2} \right)$

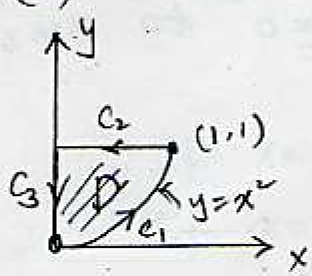
$= \boxed{\frac{8}{3}(e^4 - 1)}$

#4 sln

(a) $$\oint_C \overbrace{(e^y\cos x + e^x)}^{M}dx + \overbrace{(e^y\sin x + y^2)}^{N}dy$$

$$\underset{\substack{\uparrow \\ \text{by Green's thm}}}{=} \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint_D \underbrace{(e^y\cos x - e^y\cos x)}_{0}dxdy = 0$$

(b)



Let $C = \alpha \cup C_2 \cup C_3$, then $\oint_C Mdx+Ndy = 0$

$$\text{Now } \underbrace{\oint_C Mdx+Ndy}_{=\,0} = \int_\alpha Mdx+Ndy + \int_{C_2\cup C_3} Mdx+Ndy$$

$$\Rightarrow \int_\alpha Mdx+Ndy = -\left(\int_{C_2\cup C_3} Mdx+Ndy\right)$$

parametrize C_2 & C_3

$C_2: \begin{cases} x=t \\ y=1 \end{cases}$ | $C_3: \begin{matrix} x=0 \\ y=t \end{matrix}$

t changes from $1 \to 0$

$$= -\left[\int_{C_2} Mdx + \underbrace{Ndy}_{0\ (\because y=1)} + \int_{C_3} \underbrace{Mdx}_{0\ (\because x=0)} + Ndy\right]$$

$$= -\left[\int_{t=1}^{0}(e^1\cos t + e^t)dt + \int_{t=1}^{0}(e^t\underbrace{\sin 0}_{0} + t^2)dt\right]$$

$$= -\left[e\sin t + e^t\Big|_{t=1}^{0} + \frac{t^3}{3}\Big|_{t=1}^{0}\right]$$

$$= -\left[\left\{e\underbrace{\sin 0}_{0} + e^0 - (e\sin 1 + e)\right\} + \left\{0 - \frac{1}{3}\right\}\right]$$

$$= \boxed{e(1+\sin 1) - \frac{2}{3}}$$