

SUMS OF z-IDEALS AND SEMIPRIME IDEALS

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Abstract: If B is a ring (or module), and K is an ideal (or submodule) of B , let $B(K) = \{(a,b) \in B \times B : a-b \in K\}$. The relationship between ideals (or submodules) of B and those of $B(K)$ is examined carefully, and this construction is used to find a lattice-ordered subring of the ring $C(\mathbb{R})$ of all continuous real-valued functions on the real line \mathbb{R} with two z -ideals whose sum is not even semiprime.

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1. Introduction

In [M1], G. Mason calls an ideal I of a commutative ring B with identity element a *z-ideal* if whenever $a, b \in B$ are in the same set of maximal ideals of B and $a \in I$, then $b \in I$. It was shown in [GJ] and [R] that if B is a solid (or absolutely convex) subring of the ring of all continuous functions on a topological space, then a sum of two z -ideals of B is a z -ideal, but no example is given in any of these papers or in the more recent [M2] of a commutative ring containing two z -ideals whose sum is not a z -ideal. We supply such an example here with the aid of a construction of independent interest. In particular, if B is a ring or a module, and K is an ideal or submodule of B , we consider $A = \{(a,b) \in B \times B : a-b \in K\}$ and examine carefully the relationship between the ideals (or submodules) of B and those of A . This construction enables us also to answer a question posed in [H1], and to give a simpler version of an example given in [HP].

2. Extensions of modules and rings

Suppose R is a ring, B is either a left or a right R -module, and K is a submodule of B . Let $A = B(K) = \{(a,b) \in B \times B : a-b \in K\}$ and call A the *extension of B by K* . The following properties of $B(K)$ are easily verified.

- (1) $A = B(K)$ is a submodule of $B \times B$ and $D = \{(b,b) : b \in B\}$ is a submodule of A isomorphic to B . If B is a unital R -module, so is A .

- (2) If $B = R$ is a ring, and K is an ideal of B , then A is a subring of $B \times B$. Clearly (2) follows from (1) since the latter applies if B is also a right R -module.

Let $p: B \rightarrow A$ be defined by letting for any $b \in B$ $p(b) = (b, b)$, let $p_1(b) = (b, 0)$ and $p_2(b) = (0, b)$. If I is a (left) submodule of B , let

- (3) $I^{(1)} = \{(i+k, i) : i \in I, k \in K\}$ and let
 $I^{(2)} = \{(i, i+k) : i \in I, k \in K\}$.

Note that for $n = 1, 2$,

- (4) $I^{(n)}$ is a submodule of A , and if I is an ideal of B , then $I^{(n)}$ is an ideal of A .

1. Theorem For $n = 1, 2$, the map $I \rightarrow I^{(n)}$ is a bijection of the set of submodules of B onto the set $I^{(n)}$ of submodules of A that contain $p_n(K)$. Moreover, if $I, J \in I$, then $I^{(1)} = J^{(2)}$ if and only if $I = J \supset K$.

Proof. Suppose $n = 1$. It is clear from (3) that if $I \in I$, then $I^{(1)} \in I^{(1)}$. Suppose $S^* \in I^{(1)}$, whence $S^* \supset \{(k, 0) : k \in K\} = p_1(K)$. Let $I = \{b \in B : (b+k, b) \in S^* \text{ for some } k \in K\}$. It is routine to verify that I is a submodule of B , and, since $S^* \supset p_1(K)$, we have $S^* = I^{(1)}$. Clearly if $I, J \in I$ and $I^{(1)} = J^{(1)}$, then $I = J$, so the map $I \rightarrow I^{(1)}$ is a bijection of I onto $I^{(1)}$.

If $n = 2$, the same argument applies with a change in notation.

Suppose $I^{(1)} = J^{(2)}$. Then for any $i \in I$, there is a $j \in J$ and a $k \in K$ such that $(i, i) = (j, j+k)$. So $I \subset J$. Similarly $J \subset I$, and $I = J$. Since for every $i \in I$ and $k \in K$, there is an $i' \in I$ and $k' \in K$ such that $(i+k, i) = (i', i'+k')$, we must have $K \subset I$.

If $i \in I \setminus J$, then $(i, i) \in I^{(1)} \setminus J^{(2)}$. If $I = J$ and $k \in K \setminus I$, then $(k, k) \in I^{(1)} \setminus I^{(2)}$. This completes the proof of the theorem.

Our next result illustrates that for $n = 1, 2$, the map $I \rightarrow I^{(n)}$ of I onto $I^{(n)}$ preserves a number of algebraic properties of ideals. Recall that a ring B is called *prime* if whenever $a, b \in B$ and $aBb = \{0\}$, then $a = 0$ or $b = 0$. If the intersection of all the prime ideal of B is $\{0\}$, then B is called *semiprime*. Thus B is semiprime if $aBa = 0$ implies $a = 0$. An ideal P of a ring B is called *prime* (resp. *semiprime*) if B/P is a prime (resp. semiprime) ring. We call a ring B *reduced* if $B \neq \{0\}$ and if $b \in B$ and $b^2 = 0$ imply $b = 0$. A reduced ring B is semiprime, and the converse holds if B is commutative. For background, see [H2], [K1], or [M3].

2. Theorem. Suppose B is a ring, K is an ideal of B , $A = B(K)$, and $n = 1$ or 2 . For any proper ideal P of K

- (a) P is a prime ideal if and only if $P^{(n)}$ is prime ideal of A .
- (b) P is a semiprime ideal of B if and only if $P^{(n)}$ is a semiprime ideal of A .
- (c) If B is reduced, then P is a minimal prime ideal of B if and only if $P^{(n)}$ is a minimal prime ideal of A .
- (d) P is a maximal ideal of B if and only if $P^{(n)}$ is a maximal ideal of A .

Before giving the proof of Theorem 2, we introduce some notation. For any ring R , let $\text{Sp}(R)$ denote the family of proper prime ideals of R , $\text{MinSp}(R) = \{P \in \text{Sp}(R) : P \text{ is a minimal prime}\}$, and $\text{MaxSp}(R)$ denote the family of maximal ideals of R .

Proof. We assume that $n = 1$. If $P \in \text{Sp}(B)$, then $P \times B \in \text{Sp}(B \times B)$, so $(P \times B) \cap A = P^{(1)} \in \text{Sp}(A)$. Suppose conversely that $P^{(1)} \in \text{Sp}(A)$ for some ideal P of B , and $axb \in P$ for some $a \in P$, $b \in B \setminus P$ and all $x \in B$. Suppose $k_1, k_2, k_3 \in K$. Then $\alpha = (a+k_1, a)(x+k_2, x)(b+k_3, b) = (axb + k_4, axb)$ for some $k_4 \in K$. Thus $\alpha \in P^{(1)}$ and $b \notin P$. Hence $a \in P$, so $P \in \text{Sp}(B)$. Thus (a) holds.

Part (b) follows from a routine modification of the proof of (a). Recall from [K] that:

- (5) A prime ideal P of a reduced ring R is a minimal if and only if $a \in P$ implies there is a $b \notin P$ such that $ab = 0$. See also [HJ].

Observe that A is reduced if and only if B is. Suppose $P \in \text{MinSp}(A)$ and $\alpha = (a+k, a) \in P^{(1)}$ for some $a \in P$ and $k \in K$. Note that since B is reduced, $\{b \in B : ab = 0\} = A(a) = \{b \in B : ba = 0\}$ is a (two-sided) ideal for any $a \in B$.

We consider three cases.

(i) Assume $K \subset P$. Then since P is minimal, there are $b, c \notin P$ such that $ab = kc = 0$. Since $bc \notin P$, $\beta = (bc, bc) \notin P^{(1)}$, while $\alpha\beta = 0$. Hence $P^{(1)} \in \text{MinSp}(A)$ by (5).

(ii) Suppose that for each $a \in P$ there is a b in K but not in P such that $ab = 0$; that is assume $A(a) \cap K \not\subset P$. Then $\beta = (0, b) \in A \setminus P^{(1)}$ and $\alpha\beta = 0$. So $P^{(1)} \in \text{MinSp}(A)$ by (5).

(iii) Suppose there is an $a \in P$ such that $A(a) \cap K \subset P$. By (5) since P is minimal, $A(a) \not\subset K$. So $K \subset P$ and $P^{(1)} \in \text{MinSp}(A)$ by case (i).

Suppose conversely that $P^{(1)} \in \text{MinSp}(A)$ and $a \in P$. Since $(a, a) \in P^{(1)}$, there is a $b \notin P$ and a $k \in K$ such that $(a, a)(b+k, b) = (0, 0)$ then $ab = 0$, so $P \in \text{MinSp}(A)$ by (5). This completes the verification of (c).

Suppose $P \in \text{MaxSp}(B)$ and $a \in A \setminus P$. Then there is a $k \in K$ such that $(a+k, a)$ is not in $P^{(1)}$. For, otherwise, since $(k, 0) \in P^{(1)}$, it would follow that $(a, a) \in P^{(1)}$ contrary to the fact that $a \notin P$. Since the smallest ideal of B containing P and a is all of B , the ideal I generated by $\{(m, m) : m \in P\}$ and (a, a) contains $\{(b, b) : b \in B\}$. But $P^{(1)} \supset p_1(K)$, so $I = B$ and $P^{(1)} \in \text{MaxSp}(A)$. The proof of the converse is an exercise. This completes the proof of (d) and Theorem 2.

An element of a ring that is neither a left nor a right divisor of 0 will be called *regular*.

3. Corollary. Suppose B is a ring, K is an ideal of B , and $A = B(K)$.
- P^* is a (minimal) prime ideal of A if and only if $P^* = p^{(1)}$ or $P^* = p^{(2)}$ for some (minimal) prime ideal P of B .
 - An element (a, b) of A is regular if and only if both a and b are regular in B .

Proof. Suppose $P^* \in \text{Sp}(A)$. Since $p_1(K) \cap p_2(K) = 0$, $p_1(K) \subset P^*$ or $p_2(K) \subset P^*$. So (a) follows from Theorems 1 and 2 (a, c).

Clearly if a and b are regular in B , then (a, b) is regular in A . By (5), if (a, b) is regular, then it is in no minimal prime ideal of A . So, by Theorem 1 and part (a), neither a nor b can be in any minimal prime ideal of B . Using (5) again, we conclude that both a and b are regular in B . This completes the proof of the corollary.

4. Remarks.

(A) This argument of Theorem 2 (d) applies to maximal one-sided ideals, and this may be used to show that the (left) primitive ideals of A are of the form $p^{(j)}$ for P a primitive ideal of B . For definitions see [K1].

(B) It is well known that for any ring B with identity element, the sets of ideals, $\text{Sp}(A)$, $\text{MinSp}(B)$, and $\text{MaxSp}(B)$ are topological spaces under the Zariski (or hull-kernel) topology. It is clear from Theorem 2 and Corollary 3 that, at least if B is reduced, that $\text{Sp}(A)$, resp. $\text{MinSp}(A)$, resp. $\text{MaxSp}(A)$, is the quotient space of $\text{Sp}(B \times B)$, resp. $\text{MinSp}(B \times B)$, $\text{MaxSp}(B \times B)$ obtained by identifying $p^{(1)}$ and $p^{(2)}$ whenever P is a prime, resp. minimal prime, resp. maximal ideal of B that contains K . See for example [G].

If R is a totally-ordered ring, $B(+)$ is an abelian lattice-ordered group, and $rb \geq 0$ whenever $r \geq 0$ in R and $b \geq 0$ in B , then we call B an ℓ -module over R . Thus, every lattice-ordered abelian group is an ℓ -module over the ring \mathbb{Z} of integers with its usual order. See [BKW] for background.

A submodule I of B such that $a \in I$ and $|b| \leq |a|$ imply $b \in I$ is called *solid*. A submodule I of B is solid if and only if it is the kernel of an R -module homomorphism of B that preserves the lattice operations. If B is a lattice-ordered ring (= ℓ -ring) and I a solid submodule and an ideal, then I is called an ℓ -ideal of B . An ℓ -module over the real field is called a *Riesz space*. If $a \geq 0$ and $na \leq b$ $n = 1, 2, \dots$, imply $a = 0$, then B is said to be an *archimedean* ℓ -module. As usual, if $a \in B$, we let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. It follows that $a = a^+ - a^-$ and $|a| = a^+ + a^-$. For background material, see [BKW] and [LZ].

5. Proposition. *If B is an ℓ -module over a totally ordered ring R , and K is an ℓ -submodule of B , then $A = B(K)$ is an ℓ -module of $B \times B$ if and only if K is solid submodule of B .*

Proof. Suppose K is solid, $b \in B$ and $k \in K$. Now $|(b+k, b)| = (|b+k|, |b|)$ is in $B \times B$, $||b+k| - |b|| \leq |(b+k) - b| = |b| \leq |k| \in K$, and it follows from the solidity of K that $(|b+k|, |b|) \in A$. Hence A is an ℓ -submodule of $B \times B$.

Conversely, suppose A is an ℓ -submodule of $B \times B$ and $|\ell| \leq |k|$, where $k \in K$ and $\ell \in B$. Then

$$(6) \quad (|\ell| + |k|, |\ell|) \wedge (|k|, |k|) = (|k|, |\ell|) \in A.$$

Also $|(k, 0)| = (|k|, 0) \in A$, so $|k| \in K$. Hence $|\ell| \in K$ by (6). Since $(\ell, \ell) \vee (|\ell|, 0) = (\ell^+ + \ell^-, \ell^+) \in A$, we know that $\ell^- \in K$. Replacing ℓ by $(-\ell)$ in the last argument yields $\ell^+ \in K$. Thus $\ell \in K$ and we know that K is solid.

Generalizing a notion introduced for rings of continuous functions in [GJ], G. Mason calls an ideal I of a commutative ring R with identity element a *z-ideal* if whenever $a \in I$ and b is in every maximal ideal of R that contains a , we also have $b \in I$. See [M1] and [M2]. He notes that every *z-ideal* is semiprime and that any intersection of maximal ideals is a *z-ideal*. He calls an ideal with this latter property a *strong z-ideal*. In [M2], he uses D. Rudd's result [R] that if S is a solid subring of the ring $C(X)$ of all continuous real-valued functions on a topological space X , then a sum of two *z-ideals* of S is a *z-ideal*. Neither G. Mason nor D. Rudd give any example of a commutative ring with identity with two *z-ideals* whose sum is not a *z-ideal*. Next, we provide a large class of such examples. Recall that a ring R with identity is called *semisimple* if the intersection of all the maximal left ideals of R is $\{0\}$.

6. Proposition. Suppose B is a commutative semisimple ring with identity, K is an ideal of B , and $A = B(K)$. Then: $p_1(K)$ and $p_2(K)$ are strong z-ideals. If K fails to be semiprime, then neither is $p_1(K) + p_2(K)$; in particular this sum is not a z-ideal.

Proof. By Theorem 2 (d), if $M \in \text{MaxSp}(B)$, then $M^{(1)} = \{m+k, m : m \in M, k \in K\} \in \text{MaxSp}(A)$ and $p_1(K) = \bigcap \{M^{(1)} : M \in \text{MaxSp}(B)\}$ is an intersection of maximal ideals. Similarly, $p_2(K)$ is also a strong z-ideal. But $p_1(K) + p_2(K) = \{(k, k') : k, k' \in K\} = K^{(1)}$ fails to be semiprime by Theorem 2 (b) since K is not semiprime. Hence K is not a z-ideal.

An ℓ -ring that is a subdirect product of totally ordered rings is called an f -ring. In [HI], M. Henriksen gave a necessary and sufficient condition for a sum of two semiprime ideals of an f -ring B to be semiprime, and asked if the latter could fail to occur if B were archimedean. The following example will apply Proposition 6 to give a negative answer to this latter question.

7. Example. Let X denote a topological space such that the ring $C(X)$ of all continuous functions contains an ℓ -ideal K that is not semiprime, let $B = C(X)$, and let $A = B(K)$. (For example, we could take $X = [0, 1]$ and $K = \{f \in C(X) : |f| \leq ki \text{ for some } k \in C(X)\}$, where i is the identity function). Then A is an ℓ -ring by Proposition 5. But A is a subring of the archimedean f -ring $B \times B$ and hence is an archimedean f -ring. By Proposition 6, $p_1(K)$ and $p_2(K)$ are two strong z-ideals whose sum is not semiprime.

We close with one more application of our extension technique.

In [HP], C. Huijsmans and B. de Pagter call a subspace L of a Riesz space R a d -ideal if $a \in L$ implies $\{a\}^d \in L$, where, for any subset T of R , $T^d = \{b \in R : |b| \wedge |t| = 0 \text{ for all } t \in T\}$. They give sufficient conditions for the sum of two d -ideals to be a d -ideal, and give an example of an archimedean Riesz space where this latter fails.

We observe that Example 7 also serves this latter purpose. For, since A is reduced, T^d is the annihilator of T whenever $T \subset B$. By Theorem 2 and Corollary 3, $p_1(K)$ and $p_2(K)$ are each the intersections of all the minimal prime ideals containing it, while their sum is not even semiprime.

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