Old and New Unsolved Problems in Lattice-Ordered Rings that need not be f-Rings

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A Brief History

Recall that a *lattice-ordered ring* or *ℓ-ring* $A(+,\cdot,\vee,\wedge)$ is a set together with four binary operations such that $A(+,\cdot)$ is a ring, $A(\vee,\wedge)$ is a lattice, and letting $P = \{a \in A : a \vee 0 = a\}$, we have both $P + P$ and $P \cdot P$ contained in P. For $a \in A$, we let $a^+ = a \vee 0$, $a^- = (-a) \vee 0$, and $|a| = a \vee (-a)$. It follows that $a = a^+ - a^-$, $|a| = a^+ + a^-$, and for any $a, b \in A$, $|a+b| \leq |a| + |b|$ and $|ab| \leq |a||b|$. As usual $a \leq b$ means $(b-a) \in P$. We leave it to the reader to fill in what is meant by a lattice-ordered algebra over a totally ordered field.

Examples of such rings and algebras were studied through most of the 20th century, but the first paper to consider such rings as abstract algebraic systems was [BP56] by G. Birkhoff and R. S. Pierce published in 1956. (It was actually submitted at least two years earlier but was delayed and plagued by many typographical errors.) This pioneering paper set the stage for research in this area that continues to this day. The reader of [BP56] learns that some ℓ-rings have properties that are counter-intuitive. For example there are ℓ-algebras that are two-dimensional over the real field (with its usual order) with an identity element that fails to be positive. Not a lot was done with general ℓ-rings except to convince the reader that getting a general structure theory for them is very difficult. The authors introduce a special class they call f-rings, which in the presence of the axiom of choice – indeed, just the weaker prime ideal theorem for Boolean algebras; see [FH88] – can be described as subdirect products of totally ordered rings. They are important generalizations of function rings, and the overwhelming majority of the papers on ℓ-rings are concerned with them or others that do not differ greatly from them. (E.g., ℓ-rings in which all squares are nonnegative.) An incomplete survey of what was known about f-rings up to the mid-1990's is given in [H97]. In it, research on applications to real semi-algebraic geometry is covered only superficially.

While the theory of f-rings is both rich and interesting it excludes the ring of polynomials ordered coefficientwise and the ring of $n \times n$ matrices over a totally ordered field if $n > 1$. The problems stated below are mostly rather old and are concerned with ℓ-rings that are far from being f-rings.

J. Martínez (ed.), Ordered Algebraic Structures, 11–17.

A. What are the possible lattice orders on the real field $\mathbb{R}$?

The problem of whether there are any lattice orders on $\mathbb{R}$ other than the usual totally one is posed in [BP56]. In his 1976 publication, [Wi76], R. R. Wilson produced uncountably many distinct lattice orderings on $\mathbb{R}$ by making heavy use of Zorn's lemma. The nonconstructive nature of his techniques yields little insight into the nature of these many new lattice orderings, and it is unknown if he has exhibited all possible lattice orders on $\mathbb{R}$. In a follow up paper, [Wi80], Wilson shows how to construct similar pathological lattice orders on any formally real field. These orders, sometimes called Wilson orders, have been studied also by N. Schwartz in [S86] and R. Redfield in Section 5 of [R01], but neither of these authors attempt to answer question A.

The lattice orders on $\mathbb{R}$ constructed in [Wi76] have the property that for $a, b, c \in \mathbb{R}$:

> *if for $a \geq 0$, $a(b \vee c) = ab \vee ac$ and $a(b \wedge c) = ab \wedge ac$ then b and c are comparable.*

Actually, this holds in any lattice-ordered division ring D in which $1 > 0$.

Proof. Suppose b and c are incomparable, let $\gamma = b - b \wedge c$, and let $\delta = c - b \wedge c$. Then $\gamma > 0$ and $|\gamma^{-1}| \geq \gamma^{-1}$, so $|\gamma^{-1}|\gamma \geq \gamma\gamma^{-1} = 1 > 0$. Similarly, $|\delta^{-1}|\delta \geq 1$. So

$$(|\gamma^{-1}| \vee |\delta^{-1}|)\gamma \wedge (|\gamma^{-1}| \vee |\delta^{-1}|)\delta \geq |\gamma^{-1}|\gamma \wedge |\delta^{-1}|\delta \geq 1 > 0.$$

But $\gamma \wedge \delta = 0$. So, by assumption, for all $a \geq 0$ in D, $a\gamma \wedge a\delta = a(\gamma \wedge \delta) = 0$. Then, for $a = |\gamma^{-1}| \vee |\delta^{-1}|$, we have $a\gamma \wedge a\delta > 0$, a contradiction. Hence b and c are comparable. ∎

Some steps along the way to answering Question A would be to find out which lattice-orders on $\mathbb{R}$ are archimedean? What can be said about lattice orderings on algebraic or transcendental extensions of $\mathbb{R}$ equipped with a Wilson order? Are lattice orderings of $\mathbb{R}$ that are not total of some value, or just examples of pathology? Surely, classifying the lattice-orders on $\mathbb{R}$ is worthy of additional study.

B. Can the field of complex numbers be made into a lattice-ordered field?

This question was also posed in [BP56] after showing that this field cannot be made into a two dimensional lattice-ordered algebra over $\mathbb{R}$ endowed with its usual order. Indeed, these authors classify all of the two-dimensional lattice-ordered algebras over $\mathbb{R}$. In 1962, R. A. McHaffey published in an Iraqi journal [Mc62] a similar result for the division algebra **Qu** of real quaternions over $\mathbb{R}$. There it is shown that **Qu** of real quaternions considered as an algebra over $\mathbb{R}$ with its usual order does not admit a lattice order. Actually, it is shown that **Qu** cannot be made into a partially ordered division algebra whose positive cone has a nonempty interior. This is observed in the review [P64] by R. S. Pierce of [Mc62]. In it a typographical error is also corrected. No other proof of McHaffey's (hard to locate) result seems to appear in print, but J. Ma has obtained an alternate proofs of the result in [BP56] and [Mc62], which are available in preprint form [Ma01b]. Also, conceivably, McHaffey's conclusion need not hold if the ordering induced on $\mathbb{R}$ is not the usual one. So, Question B remains open.

A lot of the work on lattice-ordered fields and division rings uses power series techniques and apply only to those infinite dimensional over $\mathbb{R}$. For example, see [D89] and [R89]. Algebraic extensions of totally ordered fields have also been studied in [R00]. Much of this literature appears in the references in [R92].

C. Can the field $\mathbb{R}(x)$ of real rational functions be lattice-ordered while preserving the coefficientwise lattice order on the polynomial ring $\mathbb{R}[x]$?

The *coefficientwise order* on a subalgebra of the family $\mathbf{LR}[[x]]$ of all formal Laurent series $\sum_{k=-m}^{\infty} a_k x^k$, with coefficients in $\mathbb{R}$ is obtained by letting its positive cone be all such series in which $a_k \geq 0$ for all k. Then $\mathbf{LR}[[x]]$ is an ℓ-ring in which

$$\Big|\sum_{k=-m}^{\infty} a_k x^k\Big| = \sum_{k=-m}^{\infty} |a_k| x^k$$

and contains $\mathbb{R}[x]$ as a sub-ℓ-ring. By $\mathbb{R}(x)$ we mean, as usual,

$$\left\{ \frac{p(x)}{q(x)} : p(x), q(x) \in \mathbb{R}[x] \text{ and } q(x) \neq \mathbf{0} \right\},$$

added and multiplied like fractions. In [H71], I asked if the coefficientwise lattice-order on the sub-ℓ-ring $\mathbb{R}[x]$ can be extended to a lattice-order on $\mathbb{R}(x)$ and showed that a seemingly plausible method for extending this order did not work. While the description of the method given in [H71] and the reason given for its failure are basically correct, the exposition is badly garbled and especially deceptive because of a reference to an irrelevant paper. So a corrected and expanded version of part of [H71] will be given next.

Because each element of $\mathbb{R}(x)$ has an expansion into a Laurent series, there is an embedding of $\mathbb{R}(x)$ into $\mathbf{LR}[[x]]$ that induces a partial ordering onto the former. So, if $\mathbb{R}(x)$ were a sublattice of $\mathbf{LR}[[x]]$ under this induced ordering, Question C would have an affirmative answer.

Recall the identity $\cos t = \frac{1}{2}(e^{it} + e^{-it})$ and hence that

$$\begin{aligned}\sum_{k=0}^{\infty} (\cos k\alpha) x^k &= \sum_{k=0}^{\infty} \frac{1}{2}((e^{i\alpha})^k + (e^{-i\alpha})^k) x^k = \frac{1}{2}\Big[\sum_{k=0}^{\infty} (xe^{i\alpha})^k + \sum_{k=0}^{\infty} (xe^{-i\alpha})^k\Big] \\ &= \frac{1}{2}\Big[\frac{1}{1 - xe^{i\alpha}} + \frac{1}{1 - xe^{-i\alpha}}\Big] = \frac{1 - (\cos\alpha)x}{1 - (2\cos\alpha)x + x^2}\end{aligned}$$

is in $\mathbb{R}((x))$, while its absolute value$\sum_{k=0}^{\infty} |\cos k\alpha| x^k$ is not in $\mathbb{R}((x))$ if α is not a rational multiple of π.

No argument is supplied to justify this in [H71], and the reader is confused by a reference to a paper by Benzaghou that is irrelevant for this purpose. A simple argument to justify this assertion follows.

For, if this latter series were in $\mathbb{R}((x))$, then, considered as a function of a complex variable, it would have an analytic continuation outside of the unit disk and hence would have to converge at some point of the unit circle. But since α is not a rational multiple of π, the values assumed by $|\cos k\alpha|$ are dense in the unit interval $[0,1]$. Thus if $|z|=1$,

$$\lim_{k\to\infty} |\cos k\alpha||z|^k = \lim_{k\to\infty} |\cos k\alpha|$$

does not exist. So $\sum_{k=0}^{\infty} |\cos k\alpha| z^k$ does not converge at any point on the unit disk and hence $\sum_{k=0}^{\infty} |\cos k\alpha| x^k$ is not in $\mathbb{R}((x))$.

This does not supply a negative answer to Question C, which remains an open problem. Some sufficient conditions for the lattice orderability of rings of formal quotients are given in [Ma01a] and [R00]. A similar but different problem is posed in [CD69].

D. Lattice-orderings on algebras of matrices.

Because finite dimensional algebras can be represented as algebras of matrices, it important for us to know the possible lattice orderings of the algebra of $n \times n$ over totally ordered fields; in part because squares of nonzero matrices need not be positive if $n \geq 2$, in the usual entry-by-entry lattice-ordering of this algebra, f-ring techniques will not help much in this study. This may be the major reason for the lack of results on the structure of lattice-ordered rings and algebras of this sort. New techniques are needed for this purpose. I will not try to summarize all of the attempts along these lines, but will concentrate on new results by S. Steinberg, J. Ma, and P. Wojciechowski that have been successful and that open up new frontiers.

In 1966, E. Weinberg studied lattice-orders on rings of 2×2 matrices over the field of rational numbers [W66]. He produced infinitely many lattice orders on this ring, claimed that he had produced them all, and showed that the usual coordinatewise order is the only lattice-ordering in which the identity matrix is positive. In 2000, S. Steinberg showed in [St00] that Weinberg's list of lattice orders was incomplete, gave a complete listing, and extended Weinberg's results to the algebra of 2×2 matrices over any totally ordered field. The next year in [MW01a] J. Ma and P. Wojciechowski extended the Weinberg-Steinberg result by establishing the usual coordinatewise order is the only way to make the algebra of $n \times n$ matrices for $n \geq 2$ into a lattice ordered algebra over a totally ordered field, and in [MW01b] extended the Weinberg-Steinberg results classifying lattice-orders to the $n \times n$ case. Also, as noted above, in [Ma01b], J. Ma extends the use of these techniques to study finite dimensional lattice-ordered algebras over subfields of $\mathbb{R}$ with its usual order to get alternate proofs that neither the complex field nor the algebra of real quaternions can be lattice-ordered as a finite dimensional algebra over $\mathbb{R}$.

E. Structure spaces.

A great deal has been written on spaces of various kinds of ideals in f-rings, but very little on general ℓ-rings. Much of the former may be found by examining the references in [H97], and the subject is taken up quite generally in [HK91]. The approach taken in [HK91] is more likely to be applicable to general ℓ-rings than earlier work. A substantial step forward in the study of structure spaces has been made by J. Ma and P. Wojciechowski in [MW02] where they generalize a theorem of H. Subramanian [Su68] that applies only to commutative f-rings as follows. Suppose R and S are two ℓ-rings with strictly positive identity elements whose only nilpotent element is 0, and such that the intersection of their maximal ℓ-ideals is zero. Then an ℓ-group isomorphism of R onto S that sends the identity element of R onto the identity element of S induces a homeomorphism between their spaces of maximal ℓ-ideals in the hull-kernel topology.

This opens up the door to generalizing many theorems on structure spaces on f-rings to structure spaces on more general ℓ-rings.

F. The future.

It is my belief that the time has come for workers in lattice-ordered rings to start paying much more attention to ℓ-rings that need not be f-rings. The most exciting new developments of this sort are the recent contributions of Ma and Wojciechowski, but they have just begun to apply a cleverly used axe to the tip of a large iceberg. In addition to the large number of problems stated above, there is a need to study lattice-ordered algebras of (positive) operators on infinite dimensional vector lattices. The best way to sample what has been done is to start with the book [AB85] entitled Positive Operators.

There is much to be done.

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References

[AB85] C. Aliprantis and O. Burkinshaw, *Positive Operators.* (1985) Academic Press, Inc., Orlando, FL.

[BP56] G. Birkhoff and R. S. Pierce, *Lattice-ordered rings.* Anais. Acad. Brasil Cien. **29** (1956), 41-69.

[CD69] P. Conrad and J. Dauns, *An embedding theorem for lattice-ordered fields.* Pacific J. Math. **30** (1969), 385–398.

[D89] J. Dauns, *Lattice ordered division rings exist.* Ord. Alg. Struct. (Curaçao, 1988), W. C. Holland & J. Martinez, eds.; (1989) Math. Appl. **55**, Kluwer Acad. Publ., Dordrecht, 229–234.

[FH88] D. Feldman and M. Henriksen, *f-Rings, subdirect products of totally ordered rings, and the prime ideal theorem.* Proc. Ned. Akad. Wetensh. **91** (1988), 121–126.

[H71] M. Henriksen, *On difficulties in embedding lattice-ordered integral domains in lattice-ordered fields.* General Topology and its Relations to Modern Analysis and Algebra III; Proceedings of the Third Prague Topological Symposium (1971), Academia, Prague; (1972) Academic Press, New York, 183-185.

[H97] M. Henriksen, *A survey of f-rings and some of their generalizations.* Ord. Alg. Struct. (Curaçao, 1995), W. C. Holland & J. Martinez, eds.; (1997) Kluwer Acad. Publ., 1-26.

[HK91] M. Henriksen and R. Kopperman, *A general theory of structure spaces with applications to spaces of prime ideals.* Alg. Universalis **28** (1991), 349-376.

[Ma01a] J. Ma, *The quotient rings of a class of lattice-ordered Ore domains.* Alg. Universalis, to appear.

[Ma01b] J. Ma, *Finite-dimensional algebras that do not admit a lattice order.* Preprint.

[Mc62] R. A. McHaffey, *A proof that the quaternions do not form a lattice-ordered algebra.* Proc. Iraqi Sci. Soc. **5** (1962), 70-71.

[MW01a] J. Ma and P. Wojciechowski, *A proof of the Weinberg conjecture.* Proc. AMS, to appear; (preprint available).

[MW01b] J. Ma and P. Wojciechowski, *Lattice orders on matrix algebras.* Submitted; (preprint available).

[MW02] J. Ma and P. Wojciechowski, *Structure spaces of maximal ℓ-ideals of lattice-ordered rings.* In these Proceedings, 261-274.

[P64] R. S. Pierce, *Review of [Mc62].* MR 27 (1964), #5706.

[R89] R. Redfield, *Constructing lattice-ordered fields and division rings.* Bull. Austral. Math. Soc. **40** (1989), 365-369.

[R92] R. Redfield, *Lattice-ordered fields as convolution algebras.* J. Algebra **153** (1992), 319-356.

[R00] R. Redfield, *Unexpected lattice-ordered quotient structures.* Ordered Algebraic Structures; Nanjing (2001); Gordon and Breach, Amsterdam.

[R01] R. Redfield, *Subfields of lattice-ordered fields that mimic maximal totally ordered subfields.* Czech. Math. J. **51 (126)** (2001), 143–161.

[S86] N. Schwartz, *Lattice-ordered fields.* Order **3** (1986), 179-194.

[St00] S. A. Steinberg, *On the scarcity of lattice-ordered matrix algebras, II.* Proc. AMS **128** (2000), 1605–1612.

[Su68] H. Subramanian, *Kaplansky's theorem for f-rings.* Math. Annalen **179** (1968), 70-73.

[W66] E. Weinberg, *On the scarcity of lattice-ordered matrix rings.* Pacific J. Math. **19** (1966), 561-571.

[Wi76] R. R. Wilson, *Lattice orderings on the real field.* Pacific J. Math. **63** (1976), 571-577.

[Wi80] R. R. Wilson, *Anti-f-rings.* Ordered Groups (Proc. Conf., Boise State Univ., Boise, Idaho, 1978), Lecture Notes in Pure and Appl. Math. **62** (1980), M. Dekker, New York, 47-51.

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