Topology Related to Rings of Real-Valued Continuous Functions. Where it has been and where it might be going

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There is no book of Genesis for the subject of the title, and even the earliest papers devoted to it depended on the work of predecessors less than completely aware that they were helping to start a new area. The first clearly recognizable contribution to this subject is M.H. Stone's monumental *Applications of Boolean rings to general topology* [St] published in 1937, and a second prize close enough to be considered a tie must be awarded to E. Čech for [Ce]. Both of these authors depended on the work of Tychonoff [T], who showed that a space is a completely regular Hausdorff space if and only if it is a closed subspace of a product of copies of the closed interval [0,1], and that this latter product is compact. What is now called the product topology in the case of infinite products appears for the first time in [T], but the fact that an arbitrary product of compact spaces is compact, which is usually called the *Tychonoff theorem*, appeared first in [Ce]. As was pointed out by A. Shields in [Sh], what we call the Stone-Čech (in the United States and the Čech-Stone in Europe) compactification βX of a completely regular Hausdorff space (now called a *Tychonoff space)* X was not considered important enough by M.H. Stone to mention in the introduction of [St], and its algebraic significance is not discussed in [Ce]. The first definite development of the maximal compactification βX as the space of maximal ideals of the algebra $C^*(X)$ of bounded real-valued continuous functions appears in [GK] in 1939. The first paper devoted completely to our subject was E. Hewitt's monumental paper [Hew1] This brilliant paper sent the subject on its true course despite being marred by a number of serious errors. At this point, Hewitt withdrew from the field, and concerned himself mostly with functional analysis. He did not re-enter the area again until 1976 in the form of a high quality paper written with three co-authors was concerned with residue class fields of $C(X)$ mod maximal ideals. See [ACCH]. This was his last contribution to the field. He made many others to both topology and analysis. See the memorials dedicated to Hewitt written by W.W. Comfort and K. Ross in Topological Commentary, a part of Topology Atlas that can be found on the internet at the url: http://at.yorku.ca/topology/.

A lot of papers were written on this subject in the 1950s, and the state of the art up until 1960 is summarized in the Gillman-Jerison [GJ] published first in 1960. Reading this excellently written book is a must for anyone wishing to do research in this area. Published now by Springer-Verlag, it remains in print after 42 years. Missing definitions in what follows may be found in [GJ]. It helped to create another wave of publication that continued more or less through the mid 1980's summarized incompletely in [Hen1] and in [V1]. In the interim, three books appeared [Wa] on the Stone-Čech compactification, [We] on realcompactness (alias Hewitt-Nachbin completeness), and [PW], while concerned primarily with H-closed extensions, contains a large amount of material pertinent to the theme of this article. A conference dedicated to the forthcoming 25th anniversary

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of the publication of [GJ] was held in 1982 in Cincinnatti Ohio. Its proceedings appeared in 1985; see [A].

Since that time, the volume of papers on rings of continuous functions in journals devoted mainly to general topology has diminished in the United States while increasing in some other countries. The scope of this article is too small to permit a more up to date survey, but will permit the mention only of a few sample papers. For example, work in this area has been done recently in Iran (see, for example, [AKR]), in Russia (see, for example, [V2] and [Z]), and in Spain (see, for example, [BM] and [GM]).

The hope that theorems in the main stream of general topology coupled be proved more easily by studying properties of rings of real-valued continuous functions is dashed by translation difficulties. Even though the ring of continuous functions on a realcompact space (one homeomorphic to a closed subspace of a product of real lines), "nice" topological properties translate into complicated ones algebraically, and vice versa. Also, the ring $C(X)$ cannot distinguish between a Tychonoff space X and its Hewitt real compactification υX, a distinction more naturally algebraic than topological. The latter is mitigated in part because metrizable and Lindelöf spaces are realcompact as are arbitrary products of realcompact spaces, but still creates difficulties. While this disappoints those mainly concerned with "pure" topology, it is intriguing to those of us fascinated by the translation process, and enables one to apply topological techniques to the study of certain kinds of ordered algebraic systems. For example, characterizing rings of all continuous functions on various classes of topological spaces within certain classes of lattice-ordered rings and algebras that are subdirect products of totally ordered rings or algebras. These are almost always called *f-rings.* An (incomplete) survey of activity in this area is given in [Hen2] where connections with other areas such as real semi-algebraic geometry, homological algebra, spectral theory, point-free topology, and non-standard analysis are given at least brief mention. In addition, it is of interest to set-theorists that answers to many questions which arise in the study of $C(X)$ seem to be independent in Zermelo-Fraenkel set theory together with the axiom of choice [ZFC]. For example, in [vDvM], it is shown that the Parovičenko characterization of $\beta\omega \setminus \omega$ depends essentially on the continuum hypothesis, and in [Wi], E.Wimmers give an exposition of how S. Shelah and W. Rudin show that whether there are P-points in $\beta\omega \setminus \omega$ cannot be determined in [ZFC].

Those of us who pursue the study of rings of real-valued continuous functions does not seem to have a happy home anywhere. The hypotheses of its theorems often seem unnatural both to topologists for the reasons given above and to commutative algebraists because they usually fail to obey "natural" chain conditions, and because any such integral domain is a field. Reducing mod a prime or maximal ideal yields either the real field or objects full of set-theoretic difficulties. Set-theorists examine questions in this area primarily as a possible source for undecidable problems. Functional analysts will not admit us to their fraternity because we ask and answer different questions about $C(X)$ from the ones they do, and are often not concerned about topologies on $C(X)$ itself.

Mathematicians usually admire depth over breadth, and few of us have the ability to prove deep theorems that require skills in several fields to understand. Despite this, aficionados of this kind of endeavor continue to persist. Going to Full Search on Math. Sci. Net and entering "Rings of continuous functions" in the slot labelled Anywhere yielded 439 entries at the end of January 2002, many of which appeared in the last decade. There are, without doubt, many more papers related to this subject to be found using different key words. So, while examining the interplay between a Tychonoff space X and the ring $C(X)$ may not yield instant fame, it will continue to fascinate enough mathematicians to keep it alive and well for some time to come.

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