COMMUNICATIONS IN ALGEBRA®
Vol. 32, No. 7, pp. 2639–2653, 2004

Combining Local and Von Neumann Regular Rings#

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ABSTRACT

All rings R considered are commutative and have an identity element. Contessa called R a VNL-*ring* if a or $1 - a$ has a Von Neumann inverse whenever $a \in R$. Sample results: Every prime ideal of a VNL-ring is contained in a unique maximal ideal. Local and Von Neumann regular rings are VNL and if the product of two rings is VNL, then both are Von Neumann regular, or one is Von Neumann regular and the other is VNL. The ring $\mathbb{Z}_n$ of integers mod n is VNL iff $(pq)^2 \nmid n$ whenever p and q are distinct primes. The ring $R[[x]]$ of formal power series over R is VNL iff R is local. The ring $C(X)$ of all continuous real-valued functions on a Tychonoff space X is VNL if and only if at most one point of X fails to be a P-point. All known VNL-rings satisfy SVNL, namely whenever the ideal generated by a (finite) subset of R is all of R, one of its members has a Von Neumann inverse. We show that a ring R is SVNL if and only if all maximal ideals of R are pure except maybe one. We show that $\prod_{\alpha \in I} R(\alpha)$ is an SVNL if and only if there exists $\alpha_0 \in I$, such that $R(\alpha_0)$ is an SVNL and for all $\alpha \in I - \{\alpha_0\}$, $R(\alpha)$ is a Von Neumann regular ring. Whether every VNL-ring is an SVNL is an open question.

#Communicated by W. Martindale.
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DOI: 10.1081/AGB-120037405 0092-7872 (Print); 1532-4125 (Online)

Key Words: Local rings; Von Neumann regular rings; Von Neumann local rings; Strongly Von Neumann local rings; P-space; Essential P-space.

Mathematics Subject Classification: Primary 16E50, 06E20; Secondary 54C.

1. INTRODUCTION

All rings considered below will be assumed to be commutative and to have an identity element **1** unless the contrary is stated explicitly. Such a ring R is called a *local ring* if it has exactly one maximal ideal, while R is said to be a *Von Neumann regular ring* or a VNR-*ring* if for each $a \in R$, there is an x in R such that $axa = a^2x = a$. More generally, an element of a ring R is said to have a *Von Neumann inverse* if there is an x with this latter property. (Note that if a has an inverse a^{-1}, we may take $x = a^{-1}$, so such an x is sometimes called a *generalized inverse of a*.) In any ring R let $vr(R)$ denote the set of elements that have a Von Neumann inverse and $nvr(R) = R \backslash vr(R)$.

1.1. The following facts about VNR-rings will be accepted as well known for general background, see Goodearl (1979) and Kaplansky (1970).

(a) While Von Neumann inverses are usually not unique, if $a \in vr(R)$, there is a unique element x' such that $ax'a = a$ and $x'ax' = x'$. Often we will write $x' = a^{(-1)}$.
(b) In VNR-rings every prime ideal is maximal. It follows immediately that the only rings that are both local and VNR are fields.

The following definition was introduced by Contessa (1984).

Definition 1.2. A ring R is said to be *Von Neumann local ring* or a VNL-*ring* if for every $a \in R$, at least one of a or $1 - a$ has a Von Neumann inverse.

She studied this class of rings and noted that every local ring and every VNR ring share this property and that every proper prime ideal of a VNL-ring is contained in a unique maximal ideal. Rings with this latter property are called PM-*rings* or *Gelfand rings*. (See for example Al-Ezeh, 1989. We will study also what appears to be a more restricted class of rings given by the following definition.) If $S \subset R$, then $\langle S \rangle$ denotes the ideal generated by S.

The following characterization of PM-rings appears in Contessa (1982).

Theorem 1.3 (Contessa)**.** *R is a* PM-*ring if and only if whenever $M \in \operatorname{Max}(R)$ and $m \in M$, there are $a, b \in R$ such that $[1 - a(1 - m)][1 - bm] = 0$.*

Definition 1.4. A ring R is said to be a *strongly Von Neumann local ring* (SVNL-*ring*) if whenever $\langle S \rangle = R$ for some nonempty $S \subset R$, at least one element of S has a Von Neumann inverse.

In this paper, we carry her study of VNL-rings further and produce many pertinent examples. We are unable to characterize VNL-rings abstractly in the sense of relating them to more familiar classes of rings, but we are able to do this in the case of SVNL-rings and we apply our characterization to a number of interesting classes including the rings of integers modulo an integer $n \geq 2$, power series and polynomial rings, and rings of real-valued continuous functions.

Some sample results: The ring $\mathbb{Z}_n$ of integers mod n is a VNL-ring if and only if n is not divisible by the square of the product of two distinct primes. The ring $R[[x]]$ of formal power series over a ring R is VNL if and only if R is VNL and R has no idempotents other than 0 and 1. The ring $C(X)$ of all continuous real-valued functions on a Tychonoff space X is VNL if and only if at most one point of X fails to be a P-point.

Within each of these classes of rings every VNL-ring is SVNL. We are unable to prove or disprove the assertion that every VNL-ring is SVNL.

2. CHARACTERIZING SVNL-RINGS; THE RINGS $\mathbb{Z}_n$

This section is devoted to characterizing a subclass of the class of VNL-rings that may or may not coincide with all of them. We apply this to describing exactly when the ring of integers mod n is a VNL-ring for $n \geq 2$.

2.1. The verifications of the following assertions are left as exercises:

(a) Any SVNL-ring is a VNL-ring.
(b) For any prime p and positive integer k, $\mathbb{Z}_{p^k}$ is a local ring, and $\mathbb{Z}_n$ is a VNR ring if n is a product of distinct primes. So 12 is the least positive integer such that $\mathbb{Z}_n$ is a VNL-ring that is neither a VNR nor a local ring.
(c) An integral domain R is VNL-ring if and only if for every $a \in R$, a is a unit or $1 - a$ is a unit. Thus an integral domain is a VNL-ring if and only if it is a local ring. Hence if R is a VNL-ring, then it is a PM-ring.
(d) The homomorphic image of an SVNL-ring (VNL-ring) R is SVNL-ring (VNL-ring). In particular for each proper ideal I of R, the quotient ring R/I is an SVNL-ring (VNL-ring).

2.2. Definitions, Notational Conventions, and Remarks.

(a) If $S \subset R$, then the *annihilator* $A(S)$ of $S = \{x \in R : xS = \{0\}\}$, and if $\{s\}$ is a singleton, let $A(s) = A(\{s\})$. $\mathrm{Max}(R)$ denotes the set of maximal ideals of R, The *Jacobson radical* $J(R)$ of R is the intersection of the elements of $\mathrm{Max}(R)$ and is given by $\{a \in R : (1 - ax)$ is invertible for all $x \in R\}$, $nil(R)$ denotes the ideal of nilpotent elements of R.
(b) If $I \subset R$ is an ideal, then I^* will denote $I \backslash \{0\}$. Note that $nil(R)^* \subset J(R)^* \subset nvr(R)$ and that each of these inclusions can be proper.
(c) If I is an ideal of R, let $mI = \{a \in R : I + A(a) = R\} = \{a \in R : a = ai$ for some $i \in I\}$. An ideal I is called *pure* if $I = mI$.

Some authors use the notation O_I in place of our mI (See, for example, Al-Ezeh, 1989). The equalities in (c) are easily established and a proof appears in Jenkins and McKnight (1962).

Lemma 2.3. *If $a \in R$, then the ideal $\langle a \rangle$ is pure if and only if $\langle a \rangle + A(a) = R$.*

Proof. Suppose that $\langle a \rangle$ is pure, then there exists $x = ar \in \langle a \rangle$ such that $a = ax = a^2r$. So $a(1 - ar) = 0$ which implies that $1 - ar \in A(a)$. Hence $1 = ar + 1 - ar \in \langle a \rangle + A(a)$.

Conversely, assume that $\langle a \rangle + A(a) = R$. Then there exist $r \in R, b \in A(a)$ such that $ar + b = 1$. Multiply both sides by a to get $a^2r = a$. If $c = ay \in \langle a \rangle$, then $c = ay = (a^2r)y = (ay)(ar) = c(ar)$. Hence $\langle a \rangle$ is pure. □

Theorem 2.4. *Suppose $a \in R$. Then a has a Von Neumann inverse if and only if for each maximal ideal M, $a \in M$ implies $a \in mM$.*

Proof. Suppose that $a = a^2b$ for some $b \in R$. If $M \in \mathrm{Max}(R)$ is such that $a \in M$, then $a = a^2b = a(ab) \in mM$. Conversely, suppose that for each maximal ideal $M, a \in M$ implies $a \in mM$. If for such an $a, A(a) \subset M$ as well, then $a(1 - m) = 0$ for some $m \in M$, in which case $(1 - m) \in M$. Hence $\langle a \rangle + A(a)$ is not contained in any maximal ideal and therefore $\langle a \rangle + A(a) = R$. So by Lemma 2.3, $\langle a \rangle$ is a pure ideal. Hence a has a Von Neumann inverse. □

The following corollary generalizes the result of Al-Ezeh (1989) which he proved for reduced PM-rings.

Corollary 2.5. *A ring R is* VNR *if and only if all maximal ideals of R are pure.*

Proof. It follows from Theorem 2.4 that $nvr(R) = \emptyset$ if and only if $M = mM$ for each $M \in \mathrm{Max}(R)$ and hence the result. □

The promised characterization of SVNL-rings follows.

Theorem 2.6. *The following statements are equivalent*:

(a) *All maximal ideals of R except may be one are pure.*
(b) *There exists $N \in \mathrm{Max}(R)$ such that for all $a \notin N, a$ has a Von Neumann inverse.*
(c) *The ring R is an* SVNL*-ring.*

Proof. (a) implies (b): Suppose that there exists $N \in \mathrm{Max}(R)$ such that $M = mM$ for all $M \in \mathrm{Max}(R)\backslash\{N\}$. If $a \notin N$, then for each $M \in \mathrm{Max}(R), a \in M$ implies that $a \in mM$, so a has a Von Neumann inverse by Theorem 2.4.

(b) implies (c): Assume that there exits $N \in \mathrm{Max}(R)$ such that for each $a \notin N, a$ has a Von Neumann inverse. So if $\langle a_1, a_2, \ldots, a_n \rangle = R$, then there exists i such that $a_i \notin N$ and hence the result.

(c) implies (b): Suppose that R is an SVNL-ring. If $nvr(R) = \{a \in R : a \notin vr(R)\}$ is empty, then we are done. So assume that $nvr(R)$ is not empty and I is the ideal generated by $nvr(R)$. If $1 \in I$, then $1 = \sum_{i=1}^{n} r_i s_i$ with $s_i \in nvr(R)$ for each i. Then $R = \langle s_1, s_2, \ldots, s_n \rangle$ and so there exists i such that $s_i \notin nvr(R)$, a contradiction. Hence I is contained in a maximal ideal N and for each $a \notin N$, a has a Von Neumann inverse.

(b) implies (a): Assume that there exists $N \in \mathrm{Max}(R)$ such that for each $a \notin N$, a has a Von Neumann inverse. Let $M \in \mathrm{Max}(R) \backslash \{N\}$. Let $a \in M$, if $a \notin N$, then a has a Von Neumann inverse and so $a \in mM$. If $a \in M \cap N$, and $b \in M \backslash N$, then $b \in mM$ and $a + b \in M \backslash N$, which implies $a + b \in mM$ because $a + b$ has a Von Neumann inverse. Hence $a \in mM$ for each $a \in M$ which implies that M is a pure ideal. □

Corollary 2.7. *If R is* SVNL*-ring that is not* VNR*, there is a unique N in* $\mathrm{Max}(R)$ *containing $nvr(R)$. Moreover $nvr(R) = N \backslash mN$.*

Proof. It follows by Corollary 2.5 and Theorem 2.6 that there exists $N \in \mathrm{Max}(R)$ such that $N \neq mN$ and for $M \in \mathrm{Max}(R) \backslash \{N\}, M = mM$. So $nvr(R) = N \backslash mN \subset N$. Let $a \in N \backslash mN$ and let $M \in \mathrm{Max}(R) \backslash \{N\}$. Let $b \in M \backslash N$, since N is a maximal ideal there exists $c \in R$ such that $1 - bc \in N$. Since R is a PM-ring, there exist by Theorem 1.3, $x, y \in R$ such that $(1 - x(1 - bc))(1 - ybc) = 0$. So $(1 - ybc) = x(1 - bc)(1 - ybc)$ and hence $(1 - ybc) \in mN$ which implies that $a + 1 - ybc \in N - mN$. Hence $a \notin M$ or $a + 1 - ybc \notin M$ since otherwise b would not be in M. But $a, a + 1 - ybc \in nvr(R)$, thus $nvr(R)$ is not contained in M. □

2.8. Before applying this theorem, we review some well-known facts about the ring $\mathbb{Z}$ of integers and its homomorphic images $\mathbb{Z}_n$ for some $n \geq 2$.

(a) If r and s are relatively prime, then $\mathbb{Z}_{rs}$ is the direct product $\mathbb{Z}_r \times \mathbb{Z}_s$.
(b) If $n \geq 2$ and $n = \prod_{i=1}^{m} p_i^{k_i}$ is a prime power decomposition of n, then $\mathbb{Z}_n = \prod_{i=1}^{m} \mathbb{Z}_{p_i^{k_i}}$.
(c) $\mathbb{Z}_n$ is VNR if and only if $k_i = 1$ for $1 \leq i \leq m$; that is, if and only if p^2 does not divide n for any prime p.
(d) $\mathbb{Z}_n$ is local if and only if $m = 1$; that is, if and only if n is a prime power.

The next proposition follows immediately from Corollary 2.5 and Theorem 2.6.

Proposition 2.9. *If $n \geq 2$ and $n = \prod_{i=1}^{m} p_i^{k_i}$ is a prime power decomposition of n, then $\mathbb{Z}_n$ is a* SVNL*-ring if and only if $k_i > 1$ for at most one value of i. That is, if and only if whenever p and q are distinct primes, $(pq)^2$ fails to divide n.*

While it does not follow directly from this last proposition, it is not hard to show that this condition on n also tells us exactly when $\mathbb{Z}_n$ is a VNL-ring. More generally, we study when the SVNL and VNL properties are productive.

3. PRODUCTS, SUBRINGS, AND MISCELLANEA

It is easy to see that an arbitrary product of VNR-rings is a VNR-ring while a product of two local rings will not be a local ring unless one of them is a field. What about products of VNL-rings? Observe first that if a product $R \times R$ of a ring R with itself is a VNL-ring, then R is a VNR-ring. For, if $a \in R$ has no Von Neumann inverse, then neither $(a, 1-a)$ nor $(1,1)-(a,1-a)=(1-a,a)$ has one in $R \times R$. In particular, $\mathbb{Z}_4$ is a VNL-ring, while $\mathbb{Z}_4 \times \mathbb{Z}_4$ is not.

Recall that $a \in vr(R)$, then $a^{(-1)}$ denotes the Von Neumann inverse of a such that $aa^{(-1)}a = a$ and $a^{(-1)}aa^{(-1)} = a^{(-1)}$. Also, 1_R and $1_{R(\alpha)}$ will denote the identity elements of R and $R(\alpha)$, respectively. The next two results describe what we know about direct products of VNL and SVNL-rings.

Theorem 3.1. *If $R = \prod\{R(\alpha) : \alpha \in I\}$, then $R(\alpha)$ is a* VNL*-ring if and only if there exists $\alpha_0 \in I$ such that $R(\alpha_0)$ is a* VNL*-ring and for all $\alpha \in I - \{\alpha_0\}$, $R(\alpha)$ is a* VNR*-ring.*

Proof. Suppose that R is a VNL-ring, then $R(\alpha)$ is a VNL-ring for each $\alpha \in I$ being a homomorphic image of R. Now if every $R(\alpha)$ is VNR-ring, then we have nothing to prove. So suppose that there exists $\alpha_0 \in I$ such that $R(\alpha_0)$ is not VNR-ring and let $r(\alpha_0) \in R(\alpha_0)$ such that $r(\alpha_0)$ has no Von Neumann inverse. We want to show that for each $\alpha \in I - \{\alpha_0\}$, $R(\alpha)$ is VNR-ring.

Let $r(\alpha) \in R(\alpha), \alpha \neq \alpha_0$. Consider $r = (r(i))_{i\in I} \in R$, given by

$$r(i) = \begin{cases} r(\alpha) & i = \alpha, \\ 1_{R(\alpha_0)} - r(\alpha_0) & i = \alpha_0, \\ 1_{R(i)} & \text{elsewhere.} \end{cases}$$

Then r or $1_R - r$ has a Von Neumann inverse. But $r(\alpha_0)$ has no Von Neumann inverse, which implies that $1_R - r$ doesn't have one as well. Hence r has a Von Neumann inverse, so there exists r such that $r = r^2 r^{(-1)}$. Thus for each $\alpha, r(\alpha)$ has a Von Neumann inverse, whence each $R(\alpha)$ is a VNR-ring.

Conversely, assume that there exists $\alpha_0 \in I$ such that $R(\alpha_0)$ is VNL-ring and $R(\alpha)$ is VNR-ring for each $\alpha \in I - \{\alpha_0\}$.

Let $r = (r(i))_{i \in I} \in R$. If $r(\alpha_0)$ has a Von Neumann inverse, then for each $i \in I,\ r(i) = r(i)^2 r(i)^{(-1)}$ and so $r = r^2 r^*$ where $r^* = (r(i)^{(-1)})_{i \in I}$. If $1_{R(\alpha_0)} - r(\alpha_0)$ has a Von Neumann inverse, then for each $i \in I, 1_{R(i)} - r(i) = (1_{R(i)} - r(i))^2 \times (1_{R(i)} - r(i))^{(-1)}$. Let $r^* = ((1_{R(i)} - r(i))^{(-1)})_{i \in I}$, then $1_R - r = (1_R - r)^2 r^*$. Hence R is a VNL-ring. □

It follows immediately that the characterization of when $\mathbb{Z}_n$ is an SVNL-ring also characterizes when it is a VNL-ring. Thus $\mathbb{Z}_n$ is a SVNL-ring if and only if it is a VNL-ring.

Theorem 3.2. *If $R = \prod\{R(\alpha) : \alpha \in I\}$, then R is an* SVNL*-ring if and only if there exists $\alpha_0 \in I$ such that $R(\alpha_0)$ is an* SVNL*-ring and for all $\alpha \in I - \{\alpha_0\}, R(\alpha)$ is* VNR*-ring.*

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Proof. If R is SVNL-ring, then the result follows by the previous theorem and the fact that the homomorphic image of an SVNL-ring is an SVNL-ring. So assume that there exists $\alpha_0 \in I$ such that $R(\alpha_0)$ is an SVNL-ring and for all $\alpha \in I - \{\alpha_0\}$,$R(\alpha)$ is VNR-ring. Suppose that $\langle (r(i))_{i\in I}^{(1)}, (r(i))_{i\in I}^{(2)}, \ldots, (r(i))_{i\in I}^{(n)} \rangle = R$, then $\langle r(\alpha_0)^{(1)}, r(\alpha_0)^{(2)}, \ldots, r(\alpha_0)^{(n)} \rangle = R(\alpha_0)$. So there exists j such that $r(\alpha_0)^{(j)}$ has a Von Neumann inverse in $R(\alpha_0)$. Thus $(r(i))_{i\in I}^{(j)}$ has a Von Neumann inverse in R, so R is SVNL-ring. □

The following corollary follows directly from the previous theorem.

Corollary 3.3. *The product of a* VNR-*ring and a local ring is an* SVNL-*ring.*

Recall that a ring R is said to be *integral over a subring* S if for each $r \in R$, there is a monic polynomial $f(x) \in S[x]$ such that $f(r) = 0$. It is known that if R is VNR-ring and S is a subring of R such that R is integral over S, then S is also VNR-ring. We now establish a similar result for SVNL-rings and VNL-rings. But first we prove the following lemma.

Lemma 3.4. *Let S be a subring of R such that R is integral over S. If $s \in S$ has a Von Neumann inverse in R, then it has a Von Neumann inverse in S.*

Proof. By assumption, there exists a monic polynomial $f(x) = x^n + \sum_{k=1}^{n} a_{n-k} x^{n-k} \in S[x]$ such that $f(s^{(-1)}) = 0$. If we multiply the equation $f(s^{(-1)}) = 0$ by s^n and use the fact that $s = s^2 s^{(-1)}$, then we deduce that $(ss^{(-1)})^n \in S$. But $ss^{(-1)}$ is an idempotent, so $ss^{(-1)} \in S$. Multiply the equation $f(s^{(-1)}) = 0$ by s^{n-1} this yields that $s^{n-1}(s^{(-1)})^n \in S$. But $s^{n-1}(s^{(-1)})^n = (ss^{(-1)})^{n-1} s^{(-1)} = (ss^{(-1)}) s^{(-1)} = s(s^{(-1)})^2 = s^{(-1)}$. Therefore $s^{(-1)} \in S$. Whence s has a Von Neumann inverse in S. □

Theorem 3.5. *Let R be a* VNL-*ring and let S be a subring of R such that $1_R \in S$ and R is integral over S. Then S is a* VNL-*ring.*

Proof. Let $s \in S$. Then since R is a VNL-ring, s or $1_R - s$ has a Von Neumann inverse in R. But R is integral over S. Hence it follows by the previous lemma that s or $1_R - s$ has a Von Neumann inverse in S. Therefore S is VNL-ring. □

Theorem 3.6. *Let R be an* SVNL-*ring and let S be a subring of R such that $1_R \in S$ and R is integral over S. Then S is an* SVNL-*ring.*

Proof. Suppose that $s_1 S + s_2 S + \cdots + s_n S = S$. Then $1 = \sum_{k=1}^{n} a_i s_i$ with $a_i \in S$ for each i. So $s_1 R + s_2 R + \cdots + s_n R = R$ which implies that there exists i such that s_i has a Von Neumann inverse in R. Hence s_i has a Von Neumann inverse in S. □

3.7. The following facts are well known.

(a) If a has a Von Neumann inverse, then there is a unit u of R such that au is idempotent. (For if $x = a^{(-1)}$ and $u = 1 + x - ax$, then $a^2 u = a$. So au is idempotent and because x and ax are in exactly the same (maximal) ideals,

u is a unit.) Hence $a \in vr(R)$ if and only if there is a unit u of R such that au is idempotent.

(b) If 2 is a unit of R and $e^2 = e$, then $e = \frac{1}{2}(2e-1) + \frac{1}{2}$ is a sum of two units of R.

Hence we have:

Theorem 3.8. *Every element of a* VNL-*ring* R *in which* 2 *is a unit is a sum of no more than three units.*

Proof. If $a \in vr(R)$, then $a = (au)u^{-1}$ is a sum of two units by 3.7(a) and (b). Otherwise $(1-a) \in vr(R)$ and hence is a sum of two units, in which case a is a sum of three units. □

In a Boolean ring only 0 and 1 are sums of units, so the requirement that 2 is a unit may not be dropped from the hypothesis of the theorem.

We close this section with two additional results about VNL-rings. First we recall a result proved in Brown and McCoy (1950).

Proposition 3.9 (Brown and McCoy)**.** *If* $a \in R$ *and there is a* $y \in R$ *such that* $(a - aya) \in vr(R)$, *then* $a \in vr(R)$.

Proposition 3.10. *If* a *and* b *are in* $vr(R)$, *then so is* ab.

Proof. By 3.7(a), there are units u, v such that au and bv are idempotents, so $aubv = (ab)(uv)$ is an idempotent. □

Proposition 3.11. *For any* $a \in R, a(1-a) \in vr(R)$ *if and only if both* a *and* $1-a$ *are in* $vr(R)$.

Proof. If a and $1-a$ are in $vr(R)$, then so is $a(1-a)$ by Proposition 3.10. Because $a - a^2 = (1-a) - (1-a)^2 = a(1-a)$, by Proposition 3.9, both a and $1-a$ are in $vr(R)$ if $a(1-a)$ is. □

In the remainder of this paper, we will apply the results given above to some classes of examples.

4. POWER SERIES AND POLYNOMIAL RINGS

Let $\mathscr{E}(R)$ be the set of all idempotents of R. In this section, we study rings R in which the power series ring $R[[x]]$ is VNL-ring and we show that the polynomial ring $R[x]$ is never VNL-ring. We show also that if $\mathscr{E}(R) = \{0, 1\}$, then SVNL-rings and VNL-rings are identical.

If $u(x) = \sum_{i=0}^{\infty} u_i x^i$ is in $R[[x]]$, then $u(0)$ will denote its constant term. The next lemma is well known and can be found in many books on commutative algebra (say Hungerford, 1974).

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Lemma 4.1. (a) *$u(x)$ is invertible in $R[x]$ if and only if $u(0)$ is invertible and the coefficient of each nonzero power of x is nilpotent.*

(b) *$u(x)$ is invertible in $R[[x]]$ if and only if $u(0)$ is invertible in R.*

Lemma 4.2. *If the ring $R[[x]]$ is a* VNL-*ring, then R is a* VNL-*ring and $\mathscr{E}(R[[x]]) = \{0, 1\}$.*

Proof. The first assertion follows from the fact that the map that sends $p(x)$ to $p(0)$ is an epimorphism.

If $e^2 = e \in R$, and $(e + x) \in vr(R[[x]])$, then there is a unit u of $R[[x]]$ such that

$$(e + x)^2 u = e + x \tag{$*$}$$

Expanding $u = u(x)$ as a power series and equating coefficients of x yields

$$eu_1 + 2eu_0 = 1.$$

Multiplying both sides by e yields $e = 1$.

If instead $1 - (e + x) \in vr(r[[x]])$, let $f = 1 - e$, and proceed as above. Equation $(*)$ becomes

$$(f - x)^2 u = f - x.$$

A calculation similar to the above yields $f = 1$ and hence $e = 0$. Thus, $\mathscr{E}(R) = \{0, 1\}$.

If $a(x) \in R[[x]]$ is idempotent, then so is a_0. Thus $a(x) = e + xp(x)$ for some $p(x) \in R[[x]]$, where $e = 0$ or $e = 1$. Suppose first that $e = 1$. Then $1 + 2xp(x) + x^2[p(x)]^2 = 1 + xp(x)$, so $p(x) = -x[p(x)]^2$.

Equating the coefficients of the constant terms yields $p_0 = 0$. Equating those of x yields $p_1 = -p_0 = 0$.

Continuing inductively yields $p_k = 0$ for $k = 0, 1, 2, \ldots$ and hence that $p(x) = 0$. If $e = 0$, a similar calculation yields $p(x) = 0$ in this case as well. So $\mathscr{E}(R[[x]]) = \{0, 1\}$. □

Lemma 4.3. *The ring $R[[x]]$ is an* SVNL-*ring if and only if R is* SVNL-*ring and $\mathscr{E}(R) = \{0, 1\}$.*

Proof. If $R[[x]]$ is SVNL-ring, then so is its epimorphic image R, so the result follows from Lemma 4.2. To show the converse, suppose that $\sum_{i=1}^{n} f_i(x)R[[x]] = R[[x]]$. Then $\sum_{i=1}^{n} f_i(0)R = R$. Without loss of generality assume that $f_i(0) \neq 0$, for all i. Because R is SVNL-ring, there is a j such that $f_j(0)$ has a Von Neumann inverse and so by 3.7, there exists a unit $u \in R$ such that $f_j(0)u$ is an idempotent and $f_j(0)$ is a unit. It follows by Lemma 4.1 that $f_j(x)$ has a Von Neumann inverse and hence that $R[[x]]$ is SVNL-ring. □

Lemma 4.4. *If R is a* VNL-*ring, then the following are equivalent*:

(a) $\mathscr{E}(R) = \{0, 1\}$.
(b) *If* $0 \neq a \in R$ *is not a unit, then* $1 - a$ *is a unit.*

Proof. Assume that (a) holds and a is a nonunit that has a Von Neumann inverse. Then there is a unit u such that au is idempotent. By (a) $au = 0 = a$, so $1 - a$ is a unit. If a has no Von Neumann inverse, there is a unit v such that $(1 - a)v$ is idempotent. Using (a) again yields that $1 - a$ is a unit.

Assume next that (a) does not hold. Then there is an idempotent $e \notin \{0, 1\}$, and (b) fails with $a = e$. □

Lemma 4.5. *The ring* $R[[x]]$ *is a* VNL-*ring if and only if*:

(a) *R is a* VNL-*ring, and*
(b) $\mathscr{E}(R) = \{0, 1\}$.

Proof. Assume that (a) and (b) hold. If $a(x)$ has no Von Neumann inverse, then by Lemma 4.1, $a(0)$ is not a unit of R. So by (b) and Lemma 4.4, $1 - a(0)$ is a unit and $1 - a(x)$ has a Von Neumann inverse. Thus $R[[x]]$ is a VNL-ring. The converse follows from Lemma 4.2. □

We are now ready to characterize when the power series ring $R[[x]]$ is VNL-ring.

Theorem 4.6. *For any commutative ring R, the following statements are equivalent*:

(a) *The ring R is a local ring.*
(b) *The ring* $R[[x]]$ *is a local ring.*
(c) *The ring* $R[[x]]$ *is an* SVNL-*ring.*
(d) *The ring* $R[[x]]$ *is a* VNL-*ring.*
(e) *The ring R is an* SVNL-*ring and* $\mathscr{E}(R) = \{0, 1\}$.
(f) *The ring R is a* VNL-*ring and* $\mathscr{E}(R) = \{0, 1\}$.
(g) *For each* $a \in R, a$ *is a unit or* $1 - a$ *is a unit.*

Proof. (a) implies (b). Let M be the maximal ideal of R and let $N_1, N_2 \in \mathrm{Max}(R[[x]])$. Then $N_1 \cap R = M = N_2 \cap R$. Moreover, $N_1 = \langle N_1 \cap R, x\rangle = \langle N_2 \cap R, x\rangle = N_2$. Hence $R[[x]]$ is a local ring.

(b) implies (c). If N is the maximal ideal of $R[[x]]$ and $\sum_{i=1}^{n} f_i(x)R[[x]] = R[[x]]$, then there exists j such that $f_j(x) \notin N$ and so is a unit. Thus $f_j(x)$ has a Von Neumann inverse.

(c) implies (d) is obvious.

The equivalence of (d) and (f) is a restatement of Lemma 4.5, and that of (c) and (e) is a restatement of Lemma 4.3. Clearly (e) implies (f). So, it remains only to show that (f) implies (g) implies (a).

(f) implies (g). Let $a \in R$, if a has a Von Neumann inverse, then by 3.7, there exists a unit $u \in R$ such that au is an idempotent. So by assumption $au = 0$ or $au = 1$, which implies that $a = 0$, or a is a unit. If $1 - a$ has a Von Neumann inverse, then there exists a unit $v \in R$ such that $(1 - a)v$ is an idempotent which implies that $a = 1$ or $1 - a$ is a unit.

(g) implies (a). Let M be a maximal ideal of R and let $a \in M$. For each $b \in R, 1 - ab$ is a unit in R. Hence a belongs to the Jacobson radical of R, which implies that M is its only maximal ideal. □

Because any local VNR-ring is a field, it follows from the theorem that:

Corollary 4.7. *The ring* $R[[x]]$ *is never a* VNR-*ring.*

Corollary 4.8. *The ring* $R[x]$ *is never a* VNL-*ring.*

Proof. If the polynomial x has a Von Neumann inverse, there would be by Lemma 4.1, a unit u of R and a $p(x) \in R[x]$ such that $x(u + xp(x)) = 0$ or 1. Because x is neither a zero divisor nor a unit, this cannot happen. Similarly, $1 - x$ has no Von Neumann inverse. □

One may also use Theorem 4.6 to show that:

Corollary 4.9. *The ring* $\mathbb{Z}_n[[x]]$ *is a* VNL-*ring if and only if n is a power of a prime.*

5. WHEN IS $C(X)$ A VNL-RING?

Throughout this paper all topological spaces X are assumed to be Tychonoff spaces; i.e., subspaces of compact Hausdorff spaces and $C(X)$ will denote the algebra of continuous real-valued functions under the usual pointwise operations. For each $f \in C(X)$, let the *zeroset* of f, $Z(f) = \{x \in X : f(x) = 0\}$, and the *cozeroset coz* $(f) = X - Z(f)$. Observe that $f \in C(X)$ is a unit (i.e., is in no maximal ideal) of $C(X)$ if and only if $Z(f) = \emptyset$, and that $e^2 = e$ implies $Z(e)$ is clopen. A point $p \in X$ such that for every $f \in C(X), f(p) = 0$ implies $p \in int\, Z(f)$ is called a *P*-point, and X is called a *P-space* if each of its points is a *P*-point. We use the notation and terminology of Gillman and Jerison (1976) and a familiarity of appropriate parts of its content.

In this section, we study spaces X for which the ring $C(X)$ is an SVNL-ring and we show that $C(X)$ is a VNL-ring if and only if it is an SVNL-ring. First we recall what happens when $C(X)$ is either a local or a VNR-ring. Observe that $C(X)$ is a local ring only when X is a one-point space, and in Chapter 14 of Gillman and Jerison (1976), it is shown that:

Proposition 5.1 (Gillman and Jerison). *The following statements are equivalent*:

(a) *The ring* $C(X)$ *is a* VNR-*ring.*
(b) $Z(f)$ *is open for each* $f \in C(X)$.
(c) *The space* X *is a* P-*space.*

As usual, βX will denote the Stone–Čech compactification of X and υX its Hewitt realcompactification. If $p \in X$, we let $M^p = \{f \in C(X) : p \in cl_{\beta X} Z(f)\}$ and $O^p = \{f \in C(X) : p \in int_{\beta X}[cl_{\beta X} Z(f)]\}$. If $p \in X$, one often denotes M^p by M_p and O^p by O_p. The following facts may be found in Gillman and Jerison (1976) and Jenkins and McKnight (1962).

$C(X)$ is a PM-ring in which $\{M^p : p \in \beta X\}$ is the collection of its maximal ideals, and for each $p \in \beta X, O^p$ is the intersection of all of the prime ideals contained in M^p. Moreover, $mM^p = O^p$.

The analog of Proposition 5.1 for VNL-rings is:

Theorem 5.2. *The following statements are equivalent*:

(a) *The ring $C(X)$ is a* VNL-*ring.*
(b) *If $f \in C(X)$, then $Z(f)$ or $Z(1-f)$ is open.*
(c) *If $Z(f) \cap Z(g) = \emptyset$, then $Z(f)$ or $Z(g)$ is open.*
(d) *At most one point of X fails to be a P-point.*

Proof. (a) if and only if (b) follows immediately from the fact noted in 3.7 that the Von Neumann regular elements are precisely those that are unit multiples of idempotents.

(b) if and only if (c). If $Z(f) \cap Z(g) = \emptyset$, and $h = f^2/(f^2+g^2)$, then $h \in C(X)$ and $Z(h) = Z(f)$ and $Z(1-h) = Z(g)$. So the two statements are equivalent.

(c) implies (d). Suppose q_1 and q_2 are distinct non P-points of X contained in disjoint neighborhoods U_1 and U_2, respectively. Then a routine construction enables us to find for $i = 1, 2$ functions $f_i \in C(X)$ such that $Z(f_i) \subseteq U_i$ and $q_i \notin intZ(f_i)$. So $Z(f_1)$ and $Z(f_2)$ are disjoint zerosets of X neither of which is open. Thus, (c) and hence (a) fails.

(d) implies (a). If X is a P-space, then there is nothing to prove. So assume that there exists $p \in X$ that is not a P-point. Suppose $f \in C(X)$. If $p \notin Z(f)$, then $Z(f)$ is open because it is the zeroset of the restriction of f to the (open) P-space $(X \backslash \{p\})$. If

$$g(x) = \begin{cases} \dfrac{1}{f}(x) & x \in coz(f) \\ 0 & \text{otherwise} \end{cases}, \quad \text{then } g \in C(X) \text{ and } f = f^2 g$$

If $p \in Z(f)$, then $p \notin Z(1-f)$ and a similar argument shows that $(1-f)$ has a Von Neumann inverse if $p \notin Z(1-f)$. Hence $C(X)$ is a VNL-ring. □

Definition 5.3. If at most one point of a space X fails to be a P-point, then X is said to be *essentially a P-space.*

Lemma 5.4. *If X is essentially a P-space and has a non P-point p, $f \in C(X)$, and $p \notin Z(f)$, then f has a Von Neumann inverse.*

Proof. By assumption $Z(f) = Z(f|_{(X\backslash\{p\})})$ is a zeroset of the P-space $(X\backslash\{p\})$, and hence is an open subspace of the open subspace $(X\backslash\{p\})$ of X. It follows that $Z(f)$ is a clopen subset of X. Arguing as in the proof that (d) implies (a) in the last theorem yields that f has a Von Neumann inverse. □

Corollary 5.5. *The ring $C(X)$ is an* SVNL-*ring if and only if it is a* VNL-*ring.*

Proof. The necessity is clear. By the theorem, $C(X)$ a VNL-ring implies X is essentially a P-space. We may assume that X has a non P-point p. If $\langle f_1, \ldots, f_n\rangle = C(X)$, where $n \geq 1$ and $f_i \in C(X)$ for $1 \leq i \leq n$, then $\bigcap_{i=1}^{n} Z(f_i) = \emptyset$. So $p \notin Z(f_j)$ for some j, in which case f_j has a Von Neumann inverse by Lemma 5.4, □

We close with a list of properties of essentially P-spaces that need not be P-spaces.

Theorem 5.6. *Suppose X is essentially a P-space with a non P-point p.*

(a) *Every subspace of X is essentially a P-space.*
(b) *An open and continuous image of X is essentially a P-space.*
(c) *If X is compact, then it is the one-point compactification of an infinite discrete space.*
(d) *If βX is essentially a P-space, then $X = \beta X$.*
(e) *If $q \in (\beta X\backslash X)$, then M^q is a pure ideal.*
(f) *υX is essentially a P-space with non P-point p.*
(g) *If Y has more than one point, then $X \times Y$ is not essentially a P-space.*

Proof. (a) is evident.

(b) Suppose $\varphi : X \to Y$ is an open continuous surjection and Z_1, Z_2 are disjoint zerosets of Y. Then, because φ is continuous, $\varphi^{-1}(Z_1)$ and $\varphi^{-1}(Z_2)$ are disjoint zerosets of the essentially P-space X. By Theorem 5.2, one of them, say $\varphi^{-1}(Z_1)$ is open. Because φ is an open map, $Z_1 = \varphi[\varphi^{-1}(Z_1)]$ is open and using Theorem 5.2 again, we know that Y is essentially a P-space.

(c) If $x \neq p$ is in X, then any compact neighborhood of it that misses p is a compact P-space and hence must be finite. So x is an isolated point. The open cover of X consisting of the singletons $\{x\}$ for $x \neq p$ together with a neighborhood of p must have a finite subcover, so each neighborhood of p must be co-finite. Thus X is the one-point compactification of the discrete space $X\backslash\{p\}$.

(d) By (c) and its proof, βX and X are the one-point compactification of the discrete space $X\backslash\{p\}$.

(e) Follows immediately from Theorems 2.6 and 5.2.

(f) Holds by Theorem 5.2 because $C(X)$ and $C(\upsilon X)$ are isomorphic.

(g) If $y_1 \neq y_2$ are in Y, then (p, y_1) and (p, y_2) are two distinct non P-points of $X \times Y$, which prevents it from being essentially a P-space by Theorem 5.2. □

Recall that a space X is said to be *pseudocompact* if every $f \in C(X)$ is bounded; that is, if $\upsilon X = \beta X$. Combining (c), (d) and (f) of the theorem yields:

Corollary 5.7. *The one-point compactification of an uncountable discrete space is the only pseudocompact essentially P-space.*

6. SPECIAL VNL-RINGS

The map $e \to (1-e)$ is a bijection of $\mathscr{E}(R)$ onto itself, so $|\mathscr{E}(R)|$ is even if it is finite. Using this observation enables one to show that if $|\mathscr{E}(R)| < \infty$, then R is a product of finitely many rings with only two idempotents. With this, Theorems 3.1 and 4.6, and Corollary 3.3, we may conclude that:

Theorem 6.1. *If R is a* VNL-*ring with only a finite number of idempotents, then it is the direct product of finitely many* VNR-*rings and a local ring, and hence is an* SVNL-*ring.*

Nicholson (1973) studies rings (that need not be either commutative or have an identity) in which for any element a that fails to have a Von Neumann inverse, there is a b such that $a + b = ab = ba$. He calls them NJ-*rings*. Thus, in the context of this paper (when all rings considered are commutative and have an identity) R is an NJ-ring exactly when $a \in nvr(R)$ implies $(1-a)$ has an inverse. It is shown in Nicholson (1973) that R is an NJ-ring if and only if $a \notin J(R)$ implies $a \in vr(R)$. It follows easily that every NJ-ring R is SVNL because any ideal of an NJ-ring generated only by elements not in $vr(R)$ must be contained in $J(R)$. So clearly $\mathbb{Z}_{12}$ is an SVNL-ring that is not an NJ-ring, which shows that examining this latter concept will not help in a search for a solution to the unsolved problem with which we close:

Is there a VNL-ring that is not an SVNL-ring?

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Received December 2002
Revised March 2003