CHAPTER 4

Radial Solutions of Quasilinear Elliptic Differential Equations

Jon Jacobsen

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711, USA
E-mail: jacobsen@math.hmc.edu

Klaus Schmitt

Department of Mathematics, 155 South 1400 East, University of Utah, Salt Lake City, UT 84112-0090, USA
E-mail: schmitt@math.utah.edu

Contents

HANDBOOK OF DIFFERENTIAL EQUATIONS
Ordinary Differential Equations, volume 1
Edited by A. Cañada, P. Drábek and A. Fonda

Abstract

This paper constitutes a short survey of the subject of radial solutions for quasilinear elliptic partial differential equations where the underlying domain is either a ball, an annular region, the exterior of a ball, or the whole space. In case the dependence of the equation on the independent variable is only in the radial direction, special solutions of such equations may be sought which depend only on the radial variable and as such are solutions of a boundary value problem for an associated nonlinear ordinary differential equation.

1. Introduction

In this paper we provide a survey of several results concerning radial solutions of quasilinear partial differential equations where the independent variable is a spatial variable varying over a domain with radial symmetry, such as a ball centered at the origin, an annular domain determined by concentric spheres centered at the origin, an exterior domain exterior to a ball, or the whole space. If the equation at hand also has the property that the dependence upon the independent variable is radial, then special radial solutions of the problem at hand may be sought and it is often the case that certain solutions having special properties, in fact, must be radial solutions.

The situation is well illustrated by the very classical problem of finding the radial eigenvalues and eigenfunctions of the Laplace operator subject to zero Dirichlet boundary conditions on the unit disk in the plane. Another illustration is the following classical Liouville–Gelfand problem which is concerned with the existence of positive solutions of the equation

$$\begin{cases} \Delta u + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ and Ω is a bounded domain in $\mathbb{R}^N$. If it is the case that $\Omega = \{x \in \mathbb{R}^N\colon |x| < 1\} := B_1(0)$, then it is reasonable to ask whether Equation (1.1) has solutions which only depend upon the radial variable. It follows from the maximum principle for elliptic equations that solutions of (1.1) can only assume positive values in the interior of the domain and then it follows by the classical result of Gidas, Ni, and Nirenberg [50] that all solutions of (1.1) are radially symmetric and (1.1) is equivalent to the ordinary differential equation's boundary value problem

$$\begin{cases} u'' + \frac{N-1}{r}u' + \lambda e^u = 0, & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{1.2}$$

for the profile $u(r) = u(|x|)$. Note that the originally discrete parameter N is now allowed to vary continuously. The results of [50] are valid for much more general situations and it follows that if $f : \mathbb{R} \to \mathbb{R}$ is a suitably smooth function (e.g., Lipschitz continuous), then any positive solution of

$$\begin{cases} \Delta u + f(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

with Ω a ball, must be radially symmetric about the center of the ball and similar results hold for the case that Ω is the whole space or a suitable exterior domain. It, on the other hand fails to hold for the case that Ω is an annular domain, in which case it often may happen that radial solutions undergo symmetry breaking bifurcations (some such results will be discussed in this paper).

If it is the case that

$$\Omega = \{x \in \mathbb{R}^N\colon 0 < a < |x| < b\},$$

then radial solutions of (1.3) are solutions of the boundary value problem

$$\begin{cases} u'' + \frac{N-1}{r}u' + f(u) = 0, & r \in (a,b), \\ u(a) = u(b) = 0. \end{cases} \tag{1.4}$$

For $N = 1$ these problems are amenable to reduction of order methods, and hence may be explicitly solved. For other values of N, this is, of course, no longer the case in general and other methods must be employed to study the solution structure of a given equation. We shall give a detailed account of problems related to (1.1) and related equations, a subject that dates back to Liouville in 1853 [71]. In 1914 Bratu [15] found an explicit solution to (1.2) when $N = 2$. Numerical progress for (1.2) when $N = 3$ was made by Frank-Kamenetskii (see [40]) in his study of thermal ignition problems. Further progress for $N = 3$ was made by Chandrasekhar [20, IV: §22–27], where (1.2) appears as a model for the temperature distribution of an isothermal gas sphere in gravitational equilibrium. Gelfand [49] built upon Frank-Kamenetskii's work when $N = 3$ and used Emden's transformation to prove the existence of a value of λ for which (1.2) has infinitely many nontrivial solutions.

In 1973 Joseph and Lundgren [61] completely characterized the solution structure of (1.2) for all N and hence, because of [50] also of the corresponding problem (1.1) in the case the domain is a ball. Other related examples arise from a larger class of partial differential operators, for example the work of Clément, de Figueiredo, and Mitidieri [22], Azorero and Alonso [44], Jacobsen [57], and Jacobsen and Schmitt [59], who consider existence and multiplicity results for the model equations

$$\begin{cases} \Delta_p u + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.5}$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator [56,74] and

$$\begin{cases} S_k\big(D^2u\big) + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $S_k(D^2u)$ is the k-Hessian operator [108], defined as the sum of all principal $k \times k$ minors of the Hessian matrix D^2u. For instance $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det D^2u$, the Monge–Ampère operator.

Note that both equations are extensions of (1.1). In particular, the results of Joseph and Lundgren explain the radial case of (1.5) for $p = 2$ and of (1.6) when $k = 1$. In [22], the authors consider (among other topics) the radial case of both (1.5) for $p = N$ and (1.6) for $N = 2k$.

All of the above problems are simply special cases of the more general family of problems

$$\begin{cases} r^{-\gamma}\big(r^{\alpha}|u'|^{\beta}u'\big)' + f(\lambda, u) = 0, & r \in (0,1), \\ u > 0, & r \in (0,1), \\ u'(0) = u(1) = 0, \end{cases} \tag{1.7}$$

or

$$\begin{cases} r^{-\gamma}\left(r^{\alpha}|u'|^{\beta}u'\right)' + f(\lambda, u) = 0, & r \in (a, b), \\ u > 0, & r \in (a, b), \\ u(a) = u(b) = 0, \end{cases} \tag{1.8}$$

where certain inequalities are to be imposed on the parameters involved in the equation. Here $'$ denotes differentiation with respect to r. For instance, if $\Omega = B_1(0)$ is the unit ball, then Equation (1.7) with $f(\lambda, u) = \lambda e^u$ arises from (1.5) and (1.6) as a consequence of a priori symmetry results (see [35] for (1.6) and [7] for (1.5)). Similar problems may be posed also for exterior domain and whole space problems.

Much work has also been devoted to boundary value problems and other qualitative studies for more general differential operators of the form

$$r^{-\gamma}\left(r^{\alpha}\phi(u')\right)' + f(\lambda, u) = 0, \tag{1.9}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism of $\mathbb{R}$, with $\phi(0) = 0$. Such problems arise in a very natural way in diffusion problems where diffusion is governed by rapidly growing terms. We shall survey some such problems below.

In most of the discussion to follow the parameter α is taken to equal the parameter γ and is denoted by $N - 1$, to indicate the partial differential equation origin of the problem, where N denotes the dimension of the underlying domain. In the discussion, however, $N - 1$ may simply denote a nonnegative parameter. The equations stated above also may depend on other parameters, denoted by λ, which dependence may be in a linear or nonlinear fashion, thus this parameter may occur as a multiplicative factor or simply as a variable in the function evaluation. Should this parameter play no role in the result at hand, we simply shall suppress the dependence.

The paper is organized as follows: We first discuss boundary value problems on a ball related to the differential operator (1.9) and rely mainly on the recent work in [45–47,54, 53]. We then proceed to discuss problems on annular domains based on some work in [9,10, 25,30,29,74]. Next, we present a detailed discussion of Gelfand type problems. Following the Gelfand case study we return to general theory and present a range of related topics including some classical oscillation and nonoscillation theorems, problems for which radial solutions can undergo symmetry breaking bifurcations (relying on work in [67,68,80–82]), and problems concerning radial ground states of problems defined in all of space.

We shall denote by $\|\cdot\|$ the supremum norm in $C[a, b]$ for any interval $[a, b] \subset \mathbb{R}$.

2. Boundary value problems on a ball

2.1. *Introduction*

In this section, we consider the existence of positive solutions for the boundary value problems

$$\begin{cases} \left(r^{N-1}\phi(u')\right)' + \lambda r^{N-1} f(u) = 0, & 0 < r < R, \\ u'(0) = u(R) = 0, \end{cases} \tag{2.1}$$

where ϕ is an odd increasing homeomorphism on $\mathbb{R}$ and λ is a positive parameter (i.e., we consider the case that $\alpha = \gamma = N - 1$).

As pointed out before, Equation (2.1) arises in the study of radial solutions for quasilinear elliptic boundary value problems of the form

$$\begin{cases} \operatorname{div}\big(A(\nabla u)\big) + \lambda f(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.2}$$

where Ω is a ball of radius R in $\mathbb{R}^N$.

We first discuss existence results for (2.1) when $f(0) = 0$. These results are of bifurcation type and are based on the work [45,46]. We then discuss the cases when $f(0) < 0$ or $f(0) > 0$, and f is ϕ superlinear at ∞. In the former case there exists a positive, decreasing solution to (2.1) for λ small. The asymptotic behavior of the solution (as $\lambda \to 0$) will also discussed. This generalizes the results in [19,2] (see also [100]) in which $\phi(x) = x$, and complements those in [31,74] where similar problems were considered on an annulus. In the latter case, there exists a positive number λ^* such that (2.1) has at least two positive solutions for $\lambda < \lambda^*$, at least one for $\lambda = \lambda^*$, and none for $\lambda > \lambda^*$. This result complements corresponding results in [3,29] on annular domains. We refer to [100] for the literature on problem (2.2) with $A(x) = x$ on bounded domains. The approach used here depends on degree theory and also uses results about upper and lower solutions. Such results can be found in, for example, Lloyd [72], Berger [13], Deimling [32], and Schmitt [99].

2.2. *The case when* $f(0) = 0$

The prototypical example in this case is the partial differential equation

$$\begin{cases} \Delta u + u^p = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{2.3}$$

where $p > 1$. Note that $u = 0$ is a solution to (2.3). One approach to finding nontrivial solutions is to consider it as a perturbation of an eigenvalue equation

$$\Delta u + \lambda u + u^p = 0, \tag{2.4}$$

and study solution continua, i.e., curves in the solution set

$$\mathcal{S} = \big\{(\lambda, u)\colon\ (\lambda, u) \text{ satisfies (2.4)}\big\}.$$

For instance, if $\mathcal{S}$ contains a point $(0, u)$ with $u \neq 0$, then we obtain a nontrivial solution to (2.3). Viewing (2.1) with this motivation, we now consider the equation

$$\begin{cases} -r^{1-N}\big(r^{N-1}\phi(u')\big)' = \lambda\psi(u) + g(\lambda, u), & 0 < r < R, \\ u'(0) = u(R) = 0, \end{cases} \tag{2.5}$$

where $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$ are odd increasing homeomorphisms of $\mathbb{R}$, which both vanish at the origin. We further assume that $g(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. A solution to Equation (2.5) is a function $u \in C^1[0, R]$ with $\phi(u') \in C^1[0, R]$ which satisfies (2.5).

Furthermore we will require that ϕ, ψ satisfy the asymptotic homogeneity conditions:

$$\lim_{s \to 0} \frac{\phi(\sigma s)}{\psi(s)} = \sigma^{p-1}, \quad \text{for all } \sigma \in \mathbb{R}_+, \text{ for some } p > 1, \tag{2.6}$$

and

$$\lim_{s \to \pm\infty} \frac{\phi(\sigma s)}{\psi(s)} = \sigma^{q-1}, \quad \text{for all } \sigma \in \mathbb{R}_+, \text{ for some } q > 1. \tag{2.7}$$

We note that if the pair ϕ, ψ satisfies the asymptotic homogeneity conditions (2.6) and (2.7), then the function ϕ satisfies both of these conditions with ψ replaced by ϕ and also ψ satisfies both of these conditions with ϕ replaced by ψ. Such conditions appear in a variety of contexts (see, e.g., [95]).

Let $u(r)$ be a solution of (2.5). By integrating the equation in (2.5) we see that $u(r)$ satisfies the integral equation

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\big(\lambda\psi\big(u(\xi)\big) + g\big(\lambda, u(\xi)\big)\big)\,\mathrm{d}\xi\right] \mathrm{d}s. \tag{2.8}$$

This equation will be fundamental for much of the subsequent analysis.

2.2.1. *Index calculations.* Let us consider (2.5) with $g = 0$:

$$\begin{cases} -r^{1-N}\big(r^{N-1}\phi(u')\big)' = \lambda\psi(u), & 0 < r < R, \\ u'(0) = u(R) = 0. \end{cases} \tag{2.9}$$

A value λ such that (2.9) has a nontrivial solution is called an eigenvalue of (2.9). From (2.8) it follows that u is a solution if and only if u is a fixed point of the completely continuous operator $T^\lambda : C[0, R] \to C[0, R]$ defined by

$$T^\lambda(u)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\lambda\psi\big(u(\xi)\big)\,\mathrm{d}\xi\right] \mathrm{d}s. \tag{2.10}$$

If $\phi(t) = \psi(t) = |t|^{p-2}t$, then it is known that the eigenvalue problem (2.9) has a sequence of eigenvalues $\{\lambda_m = \lambda_m(p),\ m = 1, 2, \ldots\}$, with $\lambda_m(p) \to \infty$, as $\lambda \to \infty$, and this set has been completely described (see, e.g., [34,36] and the earlier papers [4,69]). We need the following lemma from [46]:

LEMMA 2.1. *Consider the problem* (2.9).

- *If* (2.6) *holds, then the Leray–Schauder degree of* $I - T^{\lambda}$ *is defined for* $B(0, \varepsilon)$, *(the open ball centered at zero with radius* ε *in* $C[0, R]$*), for all sufficiently small* ε. *Furthermore we have*

$$\deg_{\mathrm{LS}}\big(I - T^{\lambda}, B(0, \varepsilon), 0\big) = \begin{cases} 1, & \lambda < \lambda_1(p), \\ (-1)^m, & \lambda \in \big(\lambda_m(p), \lambda_{m+1}(p)\big). \end{cases} \tag{2.11}$$

- *If* (2.7) *holds, then the Leray–Schauder degree for* $I - T^{\lambda}$ *is defined for* $B(0, M)$, *for all sufficiently large* M, *and a similar formula to* (2.11) *holds, namely*

$$\deg_{\mathrm{LS}}\big(I - T^{\lambda}, B(0, M), 0\big) = \begin{cases} 1, & \lambda < \lambda_1(q), \\ (-1)^l, & \lambda \in \big(\lambda_l(q), \lambda_{l+1}(q)\big), \end{cases} \tag{2.12}$$

where $\{\lambda_l(q),\ l = 1, 2, \ldots\}$ *is the set of eigenvalues of* (2.9) *with* $\phi(t) = \psi(t) = |t|^{q-2}t$.

Lemma 2.1 may be applied to obtain an existence result for nontrivial solutions.

THEOREM 2.2. *Consider problem* (2.9) *and suppose that* ϕ, ψ *are odd increasing homeomorphisms of* $\mathbb{R}$ *with* $\phi(0) = 0 = \psi(0)$, *which satisfy* (2.6) *and* (2.7) *with* $p \neq q$. *Assume that for some* $j \in \mathbb{N}$, $\lambda_j(p) \neq \lambda_j(q)$ *and that* $\lambda \in (A, B)$, *where* $A = \min\{\lambda_j(p), \lambda_j(q)\}$ *and* $B = \max\{\lambda_j(p), \lambda_j(q)\}$. *Assume furthermore that* (A, B) *does not contain any other eigenvalue from* $\{\lambda_m(p)\}$ *or* $\{\lambda_m(q)\}$. *Then problem* (2.9) *has a nontrivial solution.*

PROOF. Assuming for example that $\lambda_j(q) > \lambda_j(p)$, it follows from Lemma 2.1 that

$$\deg_{\mathrm{LS}}\big(I - T^{\lambda}, B(0, \varepsilon), 0\big) = (-1)^j, \tag{2.13}$$

for $\varepsilon > 0$ small, and that

$$\deg_{\mathrm{LS}}\big(I - T^{\lambda}, B(0, M), 0\big) = (-1)^{j-1}, \tag{2.14}$$

for M large. Thus combining (2.13) and (2.14) with the excision property of the Leray–Schauder degree, we obtain that

$$\deg_{\mathrm{LS}}\big(I - T^{\lambda}, B(0, M) \setminus \overline{B(0, \varepsilon)}, 0\big) \neq 0, \tag{2.15}$$

yielding the existence of a nontrivial solution with $\varepsilon < \|u\| < M$. □

This theorem suggests the existence of a branch of solutions to (2.5) connecting $(\lambda_j(p), 0)$ with $(\lambda_j(q), \infty)$, for each $j \in \mathbb{N}$, generalizing the well-known property for the homogeneous case. We will return to the structure of the eigenvalue set in Section 2.2.4.

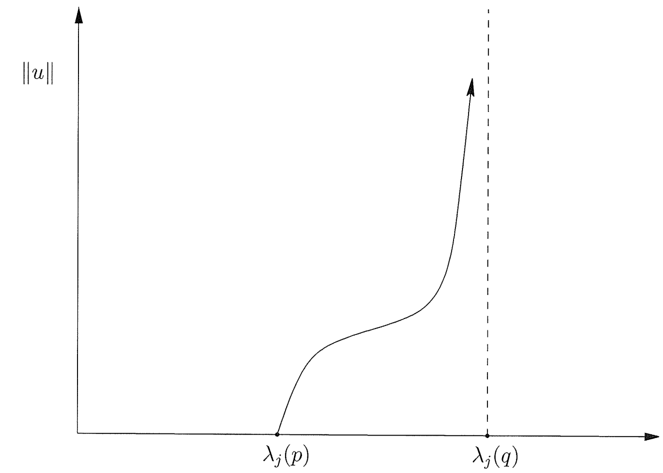

Fig. 1. Possible continua of Theorem 2.2.

2.2.2. *On initial value problems.* In this section we discuss some results for the initial value problem associated with (2.5), i.e.,

$$\begin{cases} -r^{1-N}\left(r^{N-1}\phi(u')\right)' = \lambda\psi(u) + g(\lambda, u), & 0 < r < R, \\ u(0) = d, \\ u'(0) = 0, \end{cases} \tag{2.16}$$

which will be needed in the next section. Throughout we shall assume that

$$ug(\lambda, u) \geqslant 0, \quad \text{for all } \lambda, u \in \mathbb{R}. \tag{2.17}$$

The following proposition follows from an application of the contraction mapping principle applied to (2.8):

PROPOSITION 2.3. *Suppose that $g(\lambda, u) = \mathrm{O}(|\psi(u)|)$ near zero, uniformly for r and λ in bounded intervals. Then the only solution to the problem*

$$\begin{cases} -r^{1-N}\left(r^{N-1}\phi(u')\right)' = \lambda\psi(u) + g(\lambda, u), & 0 < r < R, \\ u(r_0) = 0, \\ u'(r_0) = 0, \end{cases} \tag{2.18}$$

with $r_0 \in [0, R]$ is the trivial solution $u = 0$.

The next result concerns the oscillation of nontrivial solutions.

PROPOSITION 2.4. *Suppose that $g(\lambda, u) = \mathrm{O}(|\psi(u)|)$ near zero, uniformly for λ in bounded intervals, then nontrivial solutions of the initial value problem* (2.16) *are oscillatory, i.e., solutions are defined on all of* $[0, \infty)$ *and have infinitely many zeros.*

PROOF. Let u be a nontrivial solution of (2.16) and assume that $d > 0$. If u does not vanish in $(0, \infty)$, then u is decreasing on $(0, \infty)$. Integrating the equation from 0 to $s \in (0, r)$ we find

$$-u'(s) = \phi^{-1}\left(\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\big(\lambda\psi\big(u(\xi)\big) + g\big(\lambda, u(\xi)\big)\big)\,\mathrm{d}\xi\right). \tag{2.19}$$

Integrating (2.19) from $r/2$ to r yields

$$u(r/2) \geqslant \int_{r/2}^{r} \phi^{-1}\left(\frac{\lambda}{s^{N-1}} \int_0^s \xi^{N-1}\psi\big(u(\xi)\big)\,\mathrm{d}\xi\right)\mathrm{d}s, \tag{2.20}$$

$$\geqslant \int_{r/2}^{r} \phi^{-1}\left(\frac{\lambda}{r^{N-1}} \int_0^{r/2} \xi^{N-1}\psi\big(u(r/2)\big)\,\mathrm{d}\xi\right)\mathrm{d}s. \tag{2.21}$$

Hence

$$\frac{\phi(\frac{2}{r}u(r/2))}{\phi(u(r/2))} \geqslant \frac{\lambda}{N}\frac{r}{2^N}\frac{\psi(u(r/2))}{\phi(u(r/2))}, \tag{2.22}$$

for any $r > 0$. But for $r > 2$ the left-hand side of (2.22) is less than 1 while the right-hand side can be arbitrarily large. Hence we conclude that u must have a zero which, by Proposition 2.3, is simple. □

We remark here that by the same argument any nontrivial solution of the initial value problem

$$\begin{cases} -\big(r^{N-1}\phi(v')\big)' = r^{N-1}\lambda\psi(v) + r^{N-1}g(\lambda, v), \quad 0 < r < R, \\ v(r_0) = d, \\ v'(r_0) = 0, \end{cases} \tag{2.23}$$

must have a first isolated zero to the right of r_0 and such solutions may be continued to ∞ and are oscillatory on the whole real line.

For each $d \neq 0$, let ρ_d denote the first zero of a solution u of (2.5) such that $u(0) = d$. The next two propositions show that solutions to (2.5) have Sturm type separation properties as one has for linear equations (see, e.g., [55]):

PROPOSITION 2.5. *Suppose that $g(\lambda, u) = \mathrm{O}(|\psi(u)|)$ near zero, uniformly for λ in bounded intervals. Then, for each $R > 0$ there exists $\Lambda(R)$ such that for all $\lambda > \Lambda(R)$ and all $d \neq 0$ we have that $\rho_d \leqslant R$.*

PROPOSITION 2.6. *Let ρ_{d_j} denote the jth zero of a solution u of (2.5) such that $u(0) = d$ and suppose that $g(\lambda, u) = \mathrm{O}(|\psi(u)|)$ near zero, uniformly for λ in bounded intervals. Then, for all $L > 0$ and $j \in \mathbb{N}$ there exists $\Lambda_j(L) > 0$ such that for all $\lambda > \Lambda_j(L)$ and all $d \neq 0$ we have that $\rho_{dj} \leqslant L$.*

2.2.3. *Bifurcation of solutions.* In this section we consider bifurcation problems at zero and at infinity. See [21,32,94,64] for definitions.

Let u be a solution of problem (2.5). Then u satisfies

$$u = \mathcal{F}(\lambda, u) \tag{2.24}$$

where

$$\mathcal{F}(\lambda, u)(r) = \int_r^R \phi^{-1}\left(\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\big(\lambda\psi\big(u(\xi)\big) + g\big(\lambda, u(\xi)\big)\big)\,\mathrm{d}\xi\right)\mathrm{d}s. \tag{2.25}$$

It is clear that $\mathcal{F}: \mathbb{R} \times C[0, R] \to C[0, R]$ and it is a completely continuous operator. Concerning (2.24), the following theorem is proved by a standard argument which considers a sequence of solutions approaching a bifurcation point, appropriately renormalized to take advantage of the condition on g:

THEOREM 2.7. (i) *Suppose that* $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies* $g(\lambda, s) = \mathrm{o}(|\phi(s)|)$ *near* $s = 0$ *uniformly for* λ *in bounded intervals, and that* ϕ *and* ψ *satisfy* (2.6). *If* $(\bar{\lambda}, 0)$ *is a bifurcation point for* (2.24), *then* $\bar{\lambda} = \lambda_m(p)$, *for some* $m \in \mathbb{N}$.

(ii) *Suppose that* $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies* $g(\lambda, s) = \mathrm{o}(|\phi(s)|)$ *near infinity, uniformly for* λ *in bounded intervals, and that* ϕ *and* ψ *satisfy* (2.7). *If* $(\bar{\lambda}, \infty)$ *is a bifurcation point for* (2.24), *then* $\bar{\lambda} = \lambda_m(q)$, *for some* $m \in \mathbb{N}$.

The next theorem is the main result on bifurcation of solutions to (2.24):

THEOREM 2.8. *Suppose that* $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous and satisfies* $g(\lambda, s) = \mathrm{o}(|\phi(s)|)$ *near* $s = 0$, *uniformly for* λ *in bounded intervals.*

(i) *If* ϕ *and* ψ *satisfy* (2.6), *then for each* $k \in \mathbb{N}$ *there is a connected component* $\mathcal{S}_k \subset \mathbb{R} \times C[0, 1]$ *of the set of nontrivial solutions of* (2.5) *whose closure* $\overline{\mathcal{S}_k}$ *contains* $(\lambda_k(p), 0)$. *Moreover,* $\mathcal{S}_k$ *is unbounded in* $\mathbb{R} \times C[0, 1]$ *and if* $(\lambda, u) \in \mathcal{S}_k$, *then* u *has exactly* $k - 1$ *simple zeros in* $(0, R)$.

(ii) *If* $ug(\lambda, u) \geqslant 0$, *then there exists* $M_k \in (0, \infty)$ *such that if* $(\lambda, u) \in \mathcal{S}_k$, *then* $\lambda \leqslant M_k$.

PROOF. The proof of the existence of the connected component $\mathcal{S}_k$ such that $(\lambda_k(p), 0)$ belongs to $\overline{\mathcal{S}_k}$ and that $\mathcal{S}_k$ is unbounded or contains another bifurcation point $(\lambda_j(p), 0)$, $j \neq k$, is entirely similar to that of [94]. The fact that $(\lambda_k(p), 0) \in \overline{\mathcal{S}_k}$ and that $(\lambda, u) \in \mathcal{S}_k$ implies u has exactly $k - 1$ simple zeros in $(0, R)$ is in turn similar to that of Theorem 4.1 of [34] and uses the results of Section 2.2.2. The existence of M_k such that $\lambda \leqslant M_k$ follows directly from Proposition 2.6. □

The following will serve as examples to illustrate the above results.

THEOREM 2.9. *Suppose that* ϕ, ψ *are odd increasing homeomorphisms of* $\mathbb{R}$ *with* $\phi(0) = \psi(0) = 0$, *which satisfy* (2.6) *and* (2.7) *with* $p \neq q$ *and* $ug(\lambda, u) \geqslant 0$. *Then for any* $j \in \mathbb{N}$, *there exists a connected component S_j of the set on nontrivial solutions of* (2.5) *connecting* $(\lambda_j(p), 0)$ *to* $(\lambda_j(q), \infty)$ *such that* $(\lambda, u) \in \mathcal{S}_j$ *implies* u *has exactly* $j-1$ *simple zeros in* $(0, R)$.

PROOF. It follows from Theorem 2.8 that for any $j \in \mathbb{N}$, there exists an unbounded connected component S_j of the set on nontrivial solutions emanating from $(\lambda_j(p), 0)$ such that $(\lambda, u) \in \mathcal{S}_j$ implies u has exactly $j-1$ simple zeros in $(0, R)$. Since there is an M_j such that $(\lambda, u) \in \mathcal{S}_k$ implies that $\lambda \leqslant M_k$ and since there are no nontrivial solutions of (2.5) for $\lambda = 0$, it follows that for any $M > 0$, there is $(\lambda, u) \in \mathcal{S}_k$ such that $\|u\| > M$. Hence (ii) of Theorem 2.7 implies that as $\|u\| \to \infty$ with $(\lambda, u) \in \mathcal{S}_k$, it must be that $\lambda \to \lambda_j(q)$. □

The second application is motivated by a result of [34].

THEOREM 2.10. *Consider the problem*

$$\begin{cases} -\left(r^{N-1}\phi(u')\right)' = r^{N-1}g(u), & 0 < r < R, \\ u'(0) = u(R) = 0. \end{cases} \tag{2.26}$$

Assume ϕ *is an increasing homeomorphisms of* $\mathbb{R}$ *with* $\phi(0) = 0$ *and which satisfies* (2.6) *and* (2.7) *with* $\phi = \psi$. *Further suppose that* $g : \mathbb{R} \to \mathbb{R}$ *is continuous with* $ug(u) \geqslant 0$ *and that there exist positive integers* k *and* n, *with* $k \leqslant n$, *such that*

$$\mu := \lim_{|s| \to 0} \frac{g(s)}{\phi(s)} < \lambda_k(p) \leqslant \lambda_n(q) < \nu := \lim_{|s| \to \infty} \frac{g(s)}{\phi(s)}.$$

Then for each integer $j \in (k, n)$ *Equation* (2.26) *has a solution with exactly* $j-1$ *simple zeros in* $(0, R)$. *Thus* (2.26) *possesses at least* $n - k + 1$ *nontrivial solutions.*

2.2.4. *On principal eigenvalues.* We next present results concerning the existence of positive solutions to the problem

$$\begin{cases} -r^{1-N}\left(r^{N-1}\phi(u')\right)' = \lambda\psi(u), & 0 < r < R, \\ u'(0) = u(R) = 0, \end{cases} \tag{2.27}$$

where ϕ is an increasing homeomorphism of $\mathbb{R}$ and ψ is nondecreasing with $\phi(0) = \psi(0) = 0$. A constant λ such that (2.27) has a positive (or negative) solution is called a principal eigenvalue. Note that Theorem 2.2 established the existence of solution branches to (2.27) which may or may not correspond to positive solutions.

In the case that N is a positive integer and $\phi(u) = |u|^{p-2}u = \psi(u)$ the above problem is the problem of the existence of the principal eigenvalue of the p-Laplacian on a ball of radius R in $\mathbb{R}^N$, subject to zero Dirichlet boundary data. As such it is well understood (see, e.g., [4,14,41,43,103]).

The tools that have been used for establishing the existence of such (and higher) eigenvalues come from variational methods and are usually critical point theorems for smooth functionals defined in an appropriate Sobolev space; these methods consequently also yield theorems for the case the underlying ball domain is replaced by an arbitrary bounded domain.

Here we discuss the general case (2.27) and rely on fixed point and continuation techniques based on some work in [29,46]. For the special case $N = 1$, very detailed information is available in [34,48].

We will assume further that $N \geqslant 1$ (not necessarily an integer) and ϕ and ψ satisfy for all $x \neq 0$ and $\sigma > 0$,

$$A(\sigma) \leqslant \frac{\phi(\sigma|s|)}{\psi(|s|)} \leqslant B(\sigma), \tag{2.28}$$

where $A(\sigma)$ and $B(\sigma)$ are positive constants depending on σ only.

We introduce some notation to discuss the structure of the eigenvalue set. First, let E denote the set of all principal eigenvalues. Next, for each $d > 0$, let $\Gamma(d, R)$ be defined by

$$\Gamma(d, R) = \{\lambda > 0 \mid (2.27) \text{ has a positive solution with } u(0) = d\} \tag{2.29}$$

and set

$$\Gamma^-(d, R) = \inf \Gamma(d, R). \tag{2.30}$$

Further define

$$\Gamma_1^-(R) := \liminf_{d\to\infty} \Gamma^-(d, R), \tag{2.31}$$

$$\gamma_1^-(R) := \liminf_{d\to 0} \Gamma^-(d, R), \tag{2.32}$$

$$\lambda_1^-(R) := \inf_{d>0} \Gamma^-(d, R). \tag{2.33}$$

The main result in this section is as follows:

THEOREM 2.11. *The set $E \neq \emptyset$ and there exists a smallest $\lambda_0 > 0$ such that for $\lambda < \lambda_0$ the eigenvalue problem* (2.27) *has no nontrivial solutions. For every $d > 0$, there exists $\lambda \in E$ and a positive solution u of* (2.27) *such that $u(0) = d$. Furthermore, $\Gamma^-(d, R) > 0$ and $\Gamma_1^-(R), \gamma_1^-(R), \lambda_1^-(R)$ are all nonincreasing functions of R, and*

$$\lim_{R\to 0^+} \Gamma_1^-(R) = \lim_{R\to 0^+} \gamma_1^-(R) = \lim_{R\to 0^+} \lambda_1^-(R) = \infty,$$

$$\lim_{R\to\infty} \Gamma_1^-(R) = \lim_{R\to\infty} \gamma_1^-(R) = \lim_{R\to\infty} \lambda_1^-(R) = 0.$$

There is a result dual to Theorem 2.11 for principal eigenvalues with associated negative solutions, the set of such eigenvalues may, of course, be different from the set whose existence is asserted in Theorem 2.11.

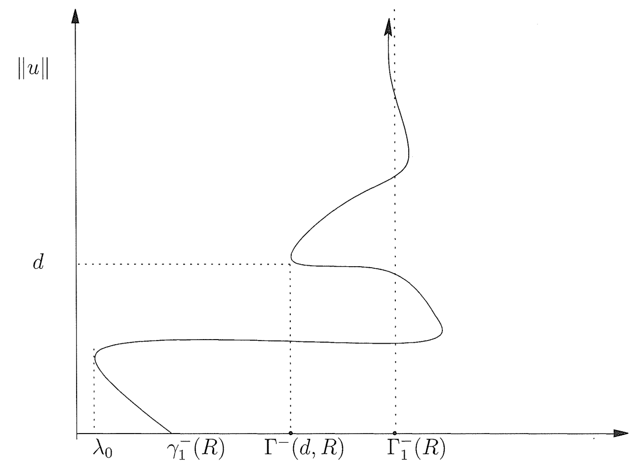

Fig. 2. Possible continua of Theorem 2.11.

We break the proof of Theorem 2.11 into several steps. First, a quick calculation shows that finding positive solutions to problem (2.27) is equivalent to finding nontrivial solutions to the problem

$$\begin{cases} \left(r^{N-1}\phi(u')\right)' + \lambda r^{N-1}\psi\left(|u|\right) = 0, & r \in (0, R), \\ u'(0) = u(R) = 0. \end{cases} \tag{2.34}$$

Let C_R denote the closed subspace of $C[0, R]$ defined by

$$C_R = \left\{u \in C[0, R] \mid u(R) = 0\right\}. \tag{2.35}$$

Then C_R is a Banach space with the inherited norm $\|\cdot\|$ from $C[0, R]$.

Similar to (2.8), we see that a solution u to (2.34) is equivalent to a fixed point of the operator

$$\mathcal{F}(\lambda, u)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi\left(\left|u(\xi)\right|\right)\mathrm{d}\xi\right]\mathrm{d}s. \tag{2.36}$$

Clearly $\mathcal{F}: [0, \infty) \times C_R \to C_R$ is a well-defined operator. Define now the operator $\mathcal{F}_\epsilon : [0, \infty) \times C_R \to C_R$, by

$$\mathcal{F}_\epsilon(\lambda, u)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\left(\psi\left(\left|u(\xi)\right|\right) + \epsilon\right)\mathrm{d}\xi\right]\mathrm{d}s, \tag{2.37}$$

where $\epsilon > 0$ is a constant. We have that $\mathcal{F}_\epsilon$ sends bounded sets of $[0, \infty) \times C_R$ into bounded sets of C_R. Moreover, $\mathcal{F}_\epsilon$ is a completely continuous operator and $\mathcal{F}_\epsilon(0, \cdot) = 0$.

Since

$$\deg_{\mathrm{LS}}\left(I - \mathcal{F}_\epsilon(0, \cdot), B(0, R), 0\right) = 1,$$

there exists a solution continuum $\mathcal{C}_\epsilon^+ \subset [0, \infty) \times C_R$ of solutions of

$$u = \mathcal{F}_\epsilon(\lambda, u) \tag{2.38}$$

with $\mathcal{C}_\epsilon^+$ unbounded in $[0, \infty) \times C_R$. In fact, this solution continuum $\mathcal{C}_\epsilon^+$ is bounded in the λ direction:

LEMMA 2.12. *There exists $\bar{\lambda} > 0$ such that if (λ, u) solve (2.27), then $\lambda \leqslant \bar{\lambda}$.*

PROOF. Let (λ, u) be a solution pair to (2.27). Thus

$$r^{N-1}\phi\big(u'(r)\big) = -\lambda \int_0^r \xi^{N-1}\big(\psi\big(|u(\xi)|\big) + \epsilon\big)\,\mathrm{d}\xi \leqslant 0$$

and

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\lambda\big(\psi\big(|u(\xi)|\big) + \epsilon\big)\,\mathrm{d}\xi\right]\mathrm{d}s \geqslant 0.$$

Hence, $u'(r) \leqslant 0$ and $u(r) \geqslant 0$ for all $r \in [0, R]$. Also

$$u(r) \geqslant \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\lambda\psi\big(|u(\xi)|\big)\,\mathrm{d}\xi\right]\mathrm{d}s.$$

Thus, for all $r \in [\frac{R}{4}, \frac{3R}{4}]$, we have that

$$u(r) \geqslant \frac{R}{4}\phi^{-1}\left[\frac{\lambda R}{N4^N}\psi\big(u(r)\big)\right]$$

or equivalently,

$$\frac{\phi(\frac{4}{R}u(r))}{\psi(u(r))} \geqslant \lambda \frac{R}{N4^N}. \tag{2.39}$$

□

Note that it follows from (2.39) and (2.28) that for all $\epsilon > 0$, sufficiently small, λ is bounded independent of ϵ. Thus, there exists ϵ_0 such that for each $d > 0$ and each $0 < \epsilon \leqslant \epsilon_0$ there exists $(\lambda_\epsilon, u_\epsilon) \in \mathcal{C}_\epsilon^+$, with $\|u_\epsilon\| = d > 0$ and $0 < \lambda_\epsilon \leqslant \bar{\lambda}$. We may now let $\epsilon \to 0$ and obtain a nontrivial solution of (2.27), with $\|u\| = d$, for some $\lambda^* \in (0, \bar{\lambda}]$.

Similar to (2.30) define

$$\Gamma^+(d, R) = \sup \Gamma(d, R).$$

It follows from the above calculations that $\Gamma^+(d, R) < \infty$. We now prove $\Gamma^-(d, R) > 0$ from Theorem 2.11:

PROPOSITION 2.13. *For each $d > 0$, $\Gamma^{-}(d, R) > 0$.*

PROOF. If not, then there exist sequences $\{u_n\}$, $\{\lambda_n\}$, with $\|u_n\| = d$ and $\lambda_n \to 0$ such that $u_n = \mathcal{F}(\lambda_n, u_n)$. The complete continuity of $\mathcal{F}$ implies $\{u_n\}$ has a convergent subsequence, with $u_{n_j} \to u^*$, and thus $u^* = \mathcal{F}(0, u^*)$. But it follows from (2.36) that $\mathcal{F}(0, u) = 0$ for all u, a contradiction. □

Paralleling our notation above, let us set

$$\Gamma_1^{+}(R) = \limsup_{d\to\infty} \Gamma^{+}(d, R), \tag{2.40}$$

$$\gamma_1^{+}(R) = \limsup_{d\to 0} \Gamma^{+}(d, R), \tag{2.41}$$

$$\lambda_1^{+}(R) = \sup_{d>0} \Gamma^{+}(d, R). \tag{2.42}$$

Then, we have:

PROPOSITION 2.14. *Under the above hypotheses on ϕ and ψ,*

$$\begin{array}{ll} \Gamma_1^{-}(R) > 0, & \Gamma_1^{+}(R) < \infty, \\ \gamma_1^{-}(R) > 0, & \gamma_1^{+}(R) < \infty, \\ \lambda_1^{-}(R) > 0, & \lambda_1^{+}(R) < \infty. \end{array}$$

PROOF. That the numbers in the second column are finite has already been discussed. To show that the numbers in the first column are positive we argue as follows. Assume (λ, u) is a solution of (2.34) with $u(0) = d$. Then

$$\|u\| = d,$$

and

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda \psi\big(|u(\xi)|\big)\,\mathrm{d}\xi\right] \mathrm{d}s. \tag{2.43}$$

Hence

$$d = u(0) \leqslant \int_0^R \phi^{-1}\left[\lambda \psi(d) \frac{s}{N}\right] \mathrm{d}s,$$

which implies

$$\frac{\phi(d/R)}{\psi(d)} \leqslant \lambda \frac{R}{N}. \tag{2.44}$$

Using condition (2.28) we obtain a lower bound for λ. □

The above proposition has the following corollary:

COROLLARY 2.15. *Let* (λ, u) *be a solution of* (2.34) *with* $u(0) = d$ *and let* $\theta \in (0, 1)$ *be fixed. Let* $r_0 \in (0, R)$ *be such that* $u(r_0) = \theta d$. *Then*

$$r_0 \geqslant \frac{N}{\lambda} A\left(\frac{1-\theta}{R}\right). \tag{2.45}$$

PROOF. Using Equation (2.43) we obtain the following

$$\theta d = \int_{r_0}^{R} \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda \psi\left(\left|u(\xi)\right|\right) \mathrm{d}\xi\right] \mathrm{d}s, \tag{2.46}$$

and hence

$$(1-\theta)d = \int_0^{r_0} \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda \psi\left(\left|u(\xi)\right|\right) \mathrm{d}\xi\right] \mathrm{d}s,$$

from which follows

$$(1-\theta)d \leqslant \int_0^{r_0} \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda \psi(d) \,\mathrm{d}\xi\right] \mathrm{d}s,$$

and

$$(1-\theta)d \leqslant R\phi^{-1}\left[\frac{\lambda\psi(d) r_0}{N}\right].$$

The result now follows from (2.28). □

The result just proved has the following consequence concerning solutions of large norm:

COROLLARY 2.16. *Let* $\{(\lambda_n, u_n)\}$ *be a sequence of solutions of* (2.34) *with* $u_n(0) = d_n$. *If* $d_n \to \infty$ *as* $n \to \infty$, *then* $u_n(r) \to \infty$ *uniformly with respect to* r *in compact subintervals of* $[0, R)$.

PROOF. Since u_n is given by

$$u_n(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \lambda_n \psi\left(\left|u_n(\xi)\right|\right) \mathrm{d}\xi\right] \mathrm{d}s,$$

for $\theta \in (0, 1)$ we obtain for $r \geqslant r_0$ (viz. Corollary 2.15) that

$$u_n(r) \geqslant \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^{r_0} \xi^{N-1} \lambda_n \psi(\theta d_n) \,\mathrm{d}\xi\right] \mathrm{d}s.$$

The conclusion follows from this inequality. □

Let us consider the initial value problem

$$\begin{cases} -\left(r^{N-1}\phi(u')\right)' = \lambda r^{N-1}\psi(u), & r > 0, \\ u(0) = d > 0, \\ u'(0) = 0. \end{cases} \tag{2.47}$$

To obtain the (local) existence of solutions of (2.47) we obtain solutions of an equivalent integral equation whose solutions are fixed points of the operator S defined by

$$S(u)(r) = d - \int_0^r \phi^{-1}\left[\frac{1}{s^{N-1}}\int_0^s \xi^{N-1}\lambda\psi\big(u(\xi)\big)\,\mathrm{d}\xi\right]\mathrm{d}s.$$

Since u is decreasing and ϕ and ψ are increasing we find that for given $\epsilon > 0$, there exists R_0 such that

$$S: \left\{u \in C[0, R_0]:\ \|u - d\| \leqslant \epsilon\right\} \to \left\{u \in C[0, R_0]:\ \|u - d\| \leqslant \epsilon\right\}$$

and that S is completely continuous. The result thus follows from Schauder's fixed point theorem. We next show that solutions of the initial value problem (2.47) exist globally and are oscillatory:

PROPOSITION 2.17. *For each $d > 0$ and each $\lambda > 0$ solutions of* (2.47) *exist globally on* $[0, \infty)$, *have only simple zeros and are oscillatory, i.e., the set of zeros is unbounded.*

PROOF. It follows from the above existence argument that a solution u of (2.47), as long as it is positive, is decreasing. If $u(r) \geqslant 0$ for $r \in [0, \infty)$, then using calculations as before, we find that for each $r > 0$

$$u(r) \geqslant r\phi^{-1}\left(\frac{\lambda r\psi(u(r))}{N2^{N-1}}\right),$$

and hence (since we may assume $r \geqslant 1$)

$$\frac{\phi(u(r))}{\psi(u(r))} \geqslant \frac{\lambda r}{N2^{N-1}},$$

which implies, using condition (2.28) that r cannot be unbounded. Therefore u must have a first zero. Easy arguments show that the zeros of u must be simple and that u is oscillatory. □

We next shall show that $\lambda_1^-(R)$, $\gamma_1^-(R)$ and $\Gamma_1^-(R)$ are nonincreasing functions of R. To this end we shall need the following elementary properties of the operator $\mathcal{F}$ defined by Equation (2.36). We note that the space C_R is a partially ordered Banach space with respect to the partial order $\leqslant$ defined by $u \leqslant v$ whenever $u(r) \leqslant v(r)$ for all $r \in [0, R]$. Further if $[u, v] = \{w \in C_R\colon\ u \leqslant w \leqslant v\}$ is an order interval in C_R, then it is a bounded closed set in C_R.

PROPOSITION 2.18. *The operator $\mathcal{F}$ defined by* (2.36) *is monotone with respect to the above partial order in $C[0, R]$ and hence in C_R and also monotone with respect to λ.*

From this proposition and the complete continuity of $\mathcal{F}$ immediately follows the following fixed point result:

PROPOSITION 2.19. *Assume there exists $[\alpha, \beta] \subset C[0, R]$ such that*

$$\mathcal{F}(\lambda, \cdot) : [\alpha, \beta] \to [\alpha, \beta].$$

Then $\mathcal{F}(\lambda, \cdot)$ has a fixed point $u \in C_R \cap [\alpha, \beta]$.

We note that the hypotheses of Proposition 2.19 will hold, whenever we can find a pair $\{\alpha, \beta\} \subset C[0, R]$ such that

$$\alpha \leqslant \beta$$

and

$$\alpha \leqslant \mathcal{F}(\lambda, \alpha),\ \mathcal{F}(\lambda, \beta) \leqslant \beta.$$

Using these facts we can now establish the following result:

THEOREM 2.20. *λ_1^-, γ_1^-, Γ_1^- are nonincreasing functions of R.*

PROOF. Assume there exist constants R_1 and R_2, with $R_1 < R_2$, such that $\lambda_1^-(R_1) < \lambda_1^-(R_2)$. Then there exists $\mu \in (\lambda_1^-(R_1), \lambda_1^-(R_2))$, such that (2.27) has a nontrivial solution $\tilde{\alpha}$ for $R = R_1$ and $\lambda = \mu$. Furthermore, there exists $\nu \geqslant \lambda_1^-(R_2)$ and a nontrivial solution β of (2.27) for $R = R_2$ and $\lambda = \nu$, with $\beta(0) = d$ as large as desired. It follows from Corollary 2.16 that for d sufficiently large $\beta(r) > \tilde{\alpha}(r)$, for each $r \in [0, R_1]$. Define

$$\alpha = \begin{cases} \tilde{\alpha}, & 0 \leqslant r \leqslant R_1, \\ 0, & R_1 \leqslant r \leqslant R_2. \end{cases}$$

Then the operator $\mathcal{F}(\mu, \cdot)$ for $R = R_2$ satisfies in the space $C[0, R_2]$

$$\alpha \leqslant \mathcal{F}(\mu, \alpha),\ \mathcal{F}(\mu, \beta) \leqslant \beta,$$

as may easily be verified. Thus by Proposition 2.19 this operator will have a fixed point in $[\alpha, \beta]$, contradicting that $\mu < \lambda_1^-(R_2)$. The monotonicity of the other functions is proved using virtually similar arguments. $\square$

REMARK 2.21.
1. Theorem 2.20 implies that problem (2.27) has no nontrivial solutions for $\lambda < \lambda_1^-(R)$.
2. Solutions of (2.27) are a priori bounded for λ in compact subintervals of $(-\infty, \Gamma_1^-(R))$.
3. The calculations and results also imply that for $\delta > 0$, sufficiently small, the degrees

$$\deg_{\mathrm{LS}}\big(I - \mathcal{F}(a, \cdot), B(0, \delta), 0\big),$$
$$\deg_{\mathrm{LS}}\big(I - \mathcal{F}(b, \cdot), B(0, \delta), 0\big)$$

are defined for $a < \gamma_1^-(R)$ and $b > \gamma_1^+(R)$ and equal 1 and 0, respectively. Hence it follows from global bifurcation theory that an unbounded continuum of positive solutions of (2.27) will bifurcate from $[\gamma_1^-(R), \gamma_1^+(R)]$.

We conclude this section with a few remarks concerning the nonhomogeneous equation

$$\begin{cases} \big(r^{N-1}\phi(u')\big)' + \lambda r^{N-1}\psi(u) = r^{N-1}h(r), & r \in (0, R), \\ u'(0) = u(R) = 0, \end{cases} \tag{2.48}$$

where $h \in L^\infty(0, R)$ is a given function. In this case, one has the following theorem:

THEOREM 2.22. *There exists* $\lambda_0 > 0$ *such that for every* $\lambda < \lambda_0$ *and every function* $h \in L^\infty(0, R)$ *the nonhomogeneous problem* (2.48) *has a solution.*

As is to be expected, λ_0 in this theorem is to the left of the set of principal eigenvalues associated with both positive and negative eigenfunctions.

PROOF. The existence of solutions to the nonhomogeneous problem is equivalent to the existence of fixed points of the operator $T(\lambda, u, h)$ defined by

$$T(\lambda, u, h)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\big(\lambda\psi\big(u(\xi)\big) - h(r)\big)\, \mathrm{d}\xi\right] \mathrm{d}s. \tag{2.49}$$

One establishes the existence of a fixed point of T in the space C_R using Proposition 2.19. To this end, we define

$$\lambda_0 = \inf\big\{\lambda\colon\ \lambda \text{ is a principal eigenvalue of (2.27)}\big\}$$

and consider problem (2.48) for values of $\lambda < \lambda_0$. Let λ be so chosen and choose μ such that $\lambda < \mu < \lambda_0$ and consider the boundary value problem:

$$\begin{cases} \big(r^{N-1}\phi(u')\big)' + \mu r^{N-1}\psi(u) = 0, & 0 < r < R, \\ u'(0) = u(R) = 0. \end{cases} \tag{2.50}$$

It then follows from the results considered earlier that there exists a solution u with $u(0) = d$ as large as we like. Then u satisfies also

$$u(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \mu\psi\big(u(\xi)\big)\, \mathrm{d}\xi\right] \mathrm{d}s.$$

Further $u(r)$ may be made arbitrarily large uniformly on compact subintervals of $[0, R]$ by choosing d sufficiently large. Hence

$$\begin{aligned} u(r) &= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} \mu\psi\big(u(\xi)\big)\, \mathrm{d}\xi\right] \mathrm{d}s \\ &= \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\big(\lambda\psi\big(u(\xi)\big) - h\big)\, \mathrm{d}\xi\right. \\ &\quad \left. + \frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\big((\mu - \lambda)\psi\big(u(\xi)\big) + h\big)\, \mathrm{d}\xi\right] \mathrm{d}s \\ &\geqslant T(\lambda, u, h)(r) \end{aligned}$$

provided

$$\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\big((\mu - \lambda)\psi\big(u(\xi)\big) + h\big)\, \mathrm{d}\xi \geqslant 0, \quad \text{for } s \in [0, R],$$

which may be accomplished for a given $h \in L^\infty(0, R)$, by choosing d sufficiently large. In a similar manner we may construct a negative α such that $\alpha \leqslant T(\lambda, \alpha, h)$. We now apply Proposition 2.19 to complete the proof. □

2.2.5. *Existence and nonexistence.* We next consider existence and nonexistence questions for the following class of problems:

$$\begin{cases} \big(r^{N-1}\phi(u')\big)' + r^{N-1} g(u) = 0, & r \in (0, R), \\ u'(0) = u(R) = 0, \end{cases} \tag{2.51}$$

where ϕ is an odd increasing homeomorphism of $\mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing continuous function such that $g(0) = 0$.

Motivated by the case of the p-Laplace operator and related work (see, e.g., [29,48]), we will assume that ϕ satisfies:

$$\limsup_{s\to\infty} \frac{\phi(\sigma s)}{\phi(s)} < \infty, \quad \text{for all } \sigma > 1 \tag{2.52}$$

and

$$\limsup_{s\to 0} \frac{\phi(\sigma s)}{\phi(s)} < \infty, \quad \text{for all } \sigma > 1. \tag{2.53}$$

Let

$$\Phi(s) = \int_0^s \phi(t)\,\mathrm{d}t, \qquad G(s) = \int_0^s g(t)\,\mathrm{d}t,$$

and denote by

$$\Gamma := \limsup_{s\to\infty} \frac{\Phi(s)}{s\phi(s)},$$

and for $\theta \in (0, 1)$,

$$\delta_\theta := \liminf_{s\to\infty} \frac{G(\theta s)}{sg(s)}.$$

The main existence result is the following:

THEOREM 2.23. *Suppose that* $\Gamma < 1$ *and*

$$\lim_{s\to 0} \frac{\phi(s)}{g(s)} = \infty, \qquad \lim_{s\to\infty} \frac{g(s)}{\phi(s)} = \infty. \tag{2.54}$$

Further let there exist $\theta \in (0, 1)$ *such that* $\delta_\theta > 0$ *and*

$$N\delta_\theta > N\Gamma - 1. \tag{2.55}$$

Then (2.51) *has at least one positive solution.*

Condition (2.55) has been used in [48] to indicate subcritical growth of g with respect to ϕ. In the case of powers, i.e., when $\phi(s) = |s|^{p-2}s$ and $g(s) = |s|^{\delta-1}s$, with $1 < p < N$ and $\delta > 0$, this condition reads

$$\delta < \frac{N(p-1)+p}{N-p}, \tag{2.56}$$

which is known to be optimal in the sense that there are no positive solutions to (2.51) if

$$\delta \geqslant \frac{N(p-1)+p}{N-p}. \tag{2.57}$$

On the other hand, if we let $\phi(s) = \ln(1+|s|)|s|^{p-2}s$ then one has a solution for $p < \delta \leqslant \frac{N(p-1)+p}{N-p}$. Thus if we allow slightly faster growth we can include equality in (2.56) [23].

In the case of general ϕ and g, the main nonexistence result is the following:

THEOREM 2.24. *If*

$$\sup_{s\geqslant 0} \frac{NG(s)}{sg(s)} \leqslant \inf_{s\geqslant 0} \frac{N\Phi(s)}{s\phi(s)} - 1, \tag{2.58}$$

then (2.51) *does not have a positive solution.*

Condition (2.55) is related to a condition that appears to have been first used by Castro and Kurepa in [18,17] for the linear operator, i.e., when $\phi(s) = s$. De Thélin and El Hachimi [37] extended the results in [17] to the p-Laplacian case by assuming that $NG(s) - \frac{N-p}{p} sg(s)$ is bounded from below for all $s \in \mathbb{R}$ and for some $\theta \in (0, 1)$,

$$NG(\theta d) - \frac{N-p}{p} dg(d) \geqslant 0, \quad \text{for } d \text{ large.} \tag{2.59}$$

We note here that although condition (2.59) seems to relax the conditions in [18], it can be proved to be equivalent when $p = 2$ and g is increasing (see [47]).

PROOF OF THEOREM 2.23. By radial symmetry, finding positive solutions to problem (2.51) is equivalent to finding nontrivial solutions to the problem

$$\begin{cases} \left(r^{N-1}\phi(u')\right)' + r^{N-1} g\left(|u|\right) = 0, & r \in (0, R), \\ u'(0) = u(R) = 0. \end{cases} \tag{2.60}$$

Recall from (2.35) that C_R is the closed subspace of $C[0, R]$ defined by those u that vanish at R. Let u be a solution of (2.60). If $T_0 : C_R \to C_R$ is defined by

$$T_0(u)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1} g\left(|u(\xi)|\right) \mathrm{d}\xi\right] \mathrm{d}s, \tag{2.61}$$

then T_0 is well defined and fixed points of T_0 will provide solutions of (2.60). Define now the operator $\mathcal{G} : [0, R] \times C_R \to C_R$, by

$$\mathcal{G}(\lambda, u)(r) = \int_r^R \phi^{-1}\left[\frac{1}{s^{N-1}} \int_0^s \xi^{N-1}\left(g\left(|u(\xi)|\right) + \lambda h\right) \mathrm{d}\xi\right] \mathrm{d}s, \tag{2.62}$$

where $h > 0$ is a constant to be fixed later. We find that $\mathcal{G}$ sends bounded sets of $[0, R] \times C_R$ into bounded sets of C_R and $\mathcal{G}(0, u) = T_0(u)$. Moreover, $\mathcal{G}$ is a completely continuous operator.

Also, let us define the operator $\mathcal{S} : [0, R] \times C_R \to C_R$,

$$\mathcal{S}(\lambda, u) = \int_r^R \phi^{-1}\left[\frac{\lambda}{s^{N-1}} \int_0^s \xi^{N-1} g\left(|u(\xi)|\right) \mathrm{d}\xi\right] \mathrm{d}s. \tag{2.63}$$

Again, we see that $\mathcal{S}$ is completely continuous and note that $\mathcal{S}(1, \cdot) = T_0$.

One proves the existence of a fixed point of T_0, and hence of a positive solution to (2.51), by using suitable a priori estimates and degree theory. Indeed, one may show that there exist $R_1 > 0$ and $\epsilon_0 > 0$ such that

$$\deg_{\mathrm{LS}}\left(I - T_0, B(0, R_1), 0\right) = 0$$