

CHAPTER 4

Radial Solutions of Quasilinear Elliptic Differential Equations

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Abstract

This paper constitutes a short survey of the subject of radial solutions for quasilinear elliptic partial differential equations where the underlying domain is either a ball, an annular region, the exterior of a ball, or the whole space. In case the dependence of the equation on the independent variable is only in the radial direction, special solutions of such equations may be sought which depend only on the radial variable and as such are solutions of a boundary value problem for an associated nonlinear ordinary differential equation.

1. Introduction

In this paper we provide a survey of several results concerning radial solutions of quasilinear partial differential equations where the independent variable is a spatial variable varying over a domain with radial symmetry, such as a ball centered at the origin, an annular domain determined by concentric spheres centered at the origin, an exterior domain exterior to a ball, or the whole space. If the equation at hand also has the property that the dependence upon the independent variable is radial, then special radial solutions of the problem at hand may be sought and it is often the case that certain solutions having special properties, in fact, must be radial solutions.

The situation is well illustrated by the very classical problem of finding the radial eigenvalues and eigenfunctions of the Laplace operator subject to zero Dirichlet boundary conditions on the unit disk in the plane. Another illustration is the following classical Liouville–Gelfand problem which is concerned with the existence of positive solutions of the equation

$$\begin{cases} \Delta u + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.1)$$

where $\lambda > 0$ and Ω is a bounded domain in \mathbb{R}^N . If it is the case that $\Omega = \{x \in \mathbb{R}^N : |x| < 1\} := B_1(0)$, then it is reasonable to ask whether Equation (1.1) has solutions which only depend upon the radial variable. It follows from the maximum principle for elliptic equations that solutions of (1.1) can only assume positive values in the interior of the domain and then it follows by the classical result of Gidas, Ni, and Nirenberg [50] that all solutions of (1.1) are radially symmetric and (1.1) is equivalent to the ordinary differential equation's boundary value problem

$$\begin{cases} u'' + \frac{N-1}{r}u' + \lambda e^u = 0, & r \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \quad (1.2)$$

for the profile $u(r) = u(|x|)$. Note that the originally discrete parameter N is now allowed to vary continuously. The results of [50] are valid for much more general situations and it follows that if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a suitably smooth function (e.g., Lipschitz continuous), then any positive solution of

$$\begin{cases} \Delta u + f(u) = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.3)$$

with Ω a ball, must be radially symmetric about the center of the ball and similar results hold for the case that Ω is the whole space or a suitable exterior domain. It, on the other hand fails to hold for the case that Ω is an annular domain, in which case it often may happen that radial solutions undergo symmetry breaking bifurcations (some such results will be discussed in this paper).

If it is the case that

$$\Omega = \{x \in \mathbb{R}^N : 0 < a < |x| < b\},$$

then radial solutions of (1.3) are solutions of the boundary value problem

$$\begin{cases} u'' + \frac{N-1}{r}u' + f(u) = 0, & r \in (a, b), \\ u(a) = u(b) = 0. \end{cases} \quad (1.4)$$

For $N = 1$ these problems are amenable to reduction of order methods, and hence may be explicitly solved. For other values of N , this is, of course, no longer the case in general and other methods must be employed to study the solution structure of a given equation. We shall give a detailed account of problems related to (1.1) and related equations, a subject that dates back to Liouville in 1853 [71]. In 1914 Bratu [15] found an explicit solution to (1.2) when $N = 2$. Numerical progress for (1.2) when $N = 3$ was made by Frank-Kamenetskii (see [40]) in his study of thermal ignition problems. Further progress for $N = 3$ was made by Chandrasekhar [20, IV: §22–27], where (1.2) appears as a model for the temperature distribution of an isothermal gas sphere in gravitational equilibrium. Gelfand [49] built upon Frank-Kamenetskii's work when $N = 3$ and used Emden's transformation to prove the existence of a value of λ for which (1.2) has infinitely many non-trivial solutions.

In 1973 Joseph and Lundgren [61] completely characterized the solution structure of (1.2) for all N and hence, because of [50] also of the corresponding problem (1.1) in the case the domain is a ball. Other related examples arise from a larger class of partial differential operators, for example the work of Clément, de Figueiredo, and Mitidieri [22], Azorero and Alonso [44], Jacobsen [57], and Jacobsen and Schmitt [59], who consider existence and multiplicity results for the model equations

$$\begin{cases} \Delta_p u + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.5)$$

where $\Delta_p = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplace operator [56,74] and

$$\begin{cases} S_k(D^2u) + \lambda e^u = 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1.6)$$

where $S_k(D^2u)$ is the k -Hessian operator [108], defined as the sum of all principal $k \times k$ minors of the Hessian matrix D^2u . For instance $S_1(D^2u) = \Delta u$ and $S_N(D^2u) = \det D^2u$, the Monge–Ampère operator.

Note that both equations are extensions of (1.1). In particular, the results of Joseph and Lundgren explain the radial case of (1.5) for $p = 2$ and of (1.6) when $k = 1$. In [22], the authors consider (among other topics) the radial case of both (1.5) for $p = N$ and (1.6) for $N = 2k$.

All of the above problems are simply special cases of the more general family of problems

$$\begin{cases} r^{-\gamma}(r^\alpha|u'|^\beta u')' + f(\lambda, u) = 0, & r \in (0, 1), \\ u > 0, & r \in (0, 1), \\ u'(0) = u(1) = 0, \end{cases} \quad (1.7)$$

or

$$\begin{cases} r^{-\gamma} (r^\alpha |u'|^\beta u')' + f(\lambda, u) = 0, & r \in (a, b), \\ u > 0, & r \in (a, b), \\ u(a) = u(b) = 0, \end{cases} \quad (1.8)$$

where certain inequalities are to be imposed on the parameters involved in the equation. Here $'$ denotes differentiation with respect to r . For instance, if $\Omega = B_1(0)$ is the unit ball, then Equation (1.7) with $f(\lambda, u) = \lambda e^u$ arises from (1.5) and (1.6) as a consequence of a priori symmetry results (see [35] for (1.6) and [7] for (1.5)). Similar problems may be posed also for exterior domain and whole space problems.

Much work has also been devoted to boundary value problems and other qualitative studies for more general differential operators of the form

$$r^{-\gamma} (r^\alpha \phi(u'))' + f(\lambda, u) = 0, \quad (1.9)$$

where $\phi: \mathbb{R} \rightarrow \mathbb{R}$ is an increasing homeomorphism of \mathbb{R} , with $\phi(0) = 0$. Such problems arise in a very natural way in diffusion problems where diffusion is governed by rapidly growing terms. We shall survey some such problems below.

In most of the discussion to follow the parameter α is taken to equal the parameter γ and is denoted by $N - 1$, to indicate the partial differential equation origin of the problem, where N denotes the dimension of the underlying domain. In the discussion, however, $N - 1$ may simply denote a nonnegative parameter. The equations stated above also may depend on other parameters, denoted by λ , which dependence may be in a linear or nonlinear fashion, thus this parameter may occur as a multiplicative factor or simply as a variable in the function evaluation. Should this parameter play no role in the result at hand, we simply shall suppress the dependence.

The paper is organized as follows: We first discuss boundary value problems on a ball related to the differential operator (1.9) and rely mainly on the recent work in [45–47, 54, 53]. We then proceed to discuss problems on annular domains based on some work in [9, 10, 25, 30, 29, 74]. Next, we present a detailed discussion of Gelfand type problems. Following the Gelfand case study we return to general theory and present a range of related topics including some classical oscillation and nonoscillation theorems, problems for which radial solutions can undergo symmetry breaking bifurcations (relying on work in [67, 68, 80–82]), and problems concerning radial ground states of problems defined in all of space.

We shall denote by $\|\cdot\|$ the supremum norm in $C[a, b]$ for any interval $[a, b] \subset \mathbb{R}$.

2. Boundary value problems on a ball

2.1. Introduction

In this section, we consider the existence of positive solutions for the boundary value problems

$$\begin{cases} (r^{N-1} \phi(u'))' + \lambda r^{N-1} f(u) = 0, & 0 < r < R, \\ u'(0) = u(R) = 0, \end{cases} \quad (2.1)$$