

The Borel-Cantelli Lemmas

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Let $\{A_n\}$ be a sequence of events. Then

$$\{A_n \text{ i.o.}\} = \limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

is the event which occurs if and only if an *infinite* number of the events A_n occur. The *i.o.* stands for “infinitely often”. Similarly,

$$\liminf_{n \rightarrow \infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

is the event which occurs if and only if *all but a finite* number of the events A_n occur. Note that

$$\liminf_{n \rightarrow \infty} A_n \subset \limsup_{n \rightarrow \infty} A_n.$$

We say that the sequence $\{A_n\}$ has a *limit* if and only if these two events are equal. One situation in which this occurs is when $\{A_n\}$ is *monotone*. In the case of monotone increasing $\{A_n\}$, the limit is $\bigcup_{n=1}^{\infty} A_n$. In the case of monotone decreasing $\{A_n\}$, the limit is $\bigcap_{n=1}^{\infty} A_n$. In any case in which $\{A_n\}$ has a *limit*, call it A , we claim that

$$\lim_{n \rightarrow \infty} P(A_n) = P(A).$$

To see this, note that

$$P(\liminf_{n \rightarrow \infty} A_n) = \lim_{n \rightarrow \infty} P(\bigcap_{k=n}^{\infty} A_k) \leq \liminf_{n \rightarrow \infty} P(A_n)$$

and

$$\limsup_{n \rightarrow \infty} P(A_n) \leq \lim_{n \rightarrow \infty} P(\bigcup_{k=n}^{\infty} A_k) = P(\limsup_{n \rightarrow \infty} A_n).$$

Direct version: If $\{A_n\}$ is a sequence of events and

$$\sum_{n=1}^{\infty} P(A_n) < \infty,$$

then

$$P(\{A_n \text{ i.o.}\}) = 0.$$

Proof:

$$\begin{aligned}
P(\{A_n \text{ i.o.}\}) &= P(\limsup_{n \rightarrow \infty} A_n) \\
&= \lim_{n \rightarrow \infty} P(\cup_{k=n}^{\infty} A_k) \\
&\leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} P(A_k)
\end{aligned}$$

and this tail sum approaches 0 as $n \rightarrow \infty$ under the assumption that the series converges.

Converse: Now suppose that $\{A_n\}$ is a sequence of *independent* events and

$$\sum_{n=1}^{\infty} P(A_n) = \infty,$$

then

$$P(\{A_n \text{ i.o.}\}) = 1.$$

Proof: First note that

$$\begin{aligned}
1 - P(\{A_n \text{ i.o.}\}) &= P(\{A_n \text{ i.o.}\}^c) \\
&= P(\liminf_{n \rightarrow \infty} A_n^c) \\
&= \lim_{n \rightarrow \infty} P(\cap_{k=n}^{\infty} A_k^c).
\end{aligned}$$

Therefore, if $P(A_n) = 1$ for an infinite number of the $\{A_n\}$, then both the hypothesis and the conclusion must hold. Consequently, we assume this is not the case and, hence, there is an n_0 such that for all $n \geq n_0$ we have $P(A_n) < 1$. We now consider only such values of n . However, because of independence of the $\{A_n\}$ and hence of the $\{A_n^c\}$,

$$\begin{aligned}
P(\cap_{k=n}^{\infty} A_k^c) &= \lim_{N \rightarrow \infty} \prod_{k=n}^N P(A_k^c) = \lim_{N \rightarrow \infty} \exp\left(\ln \prod_{k=n}^N P(A_k^c)\right) \\
&= \lim_{N \rightarrow \infty} \exp\left(\sum_{k=n}^N \ln P(A_k^c)\right) \\
&= \lim_{N \rightarrow \infty} \exp\left(\sum_{k=n}^N \ln [1 - P(A_k)]\right) \\
&\leq \lim_{N \rightarrow \infty} \exp\left(-\sum_{k=n}^N P(A_k)\right) = 0,
\end{aligned}$$

since the series diverges and the exponential function approaches 0 as the exponent approaches $-\infty$.