

Types of Convergence

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Convergence with Probability One: (as in the **strong** law of large numbers)

As $n \rightarrow \infty$, $X_n \rightarrow X$ with probability one or almost surely (a.s.)

$$\iff X_n(\omega) \rightarrow X(\omega) \text{ for } \omega \notin N, \text{ where } P(N) = 0$$

$$\iff P(\{\omega : X_n(\omega) \rightarrow X(\omega)\}) = 1$$

$$\iff P\left(\bigcup_{k=1}^{\infty} \{\omega : \text{there exists no } n(k) \text{ such that } n \geq n(k) \implies |X_n(\omega) - X(\omega)| < \frac{1}{k}\}\right) = 0$$

$$\iff \forall \varepsilon > 0, P(\{\omega : \text{there exists no } n(\varepsilon) \text{ such that } n \geq n(\varepsilon) \implies |X_n(\omega) - X(\omega)| < \varepsilon\}) = 0$$

$$\iff \forall \varepsilon > 0, P(\{\omega : \forall m, \exists n \geq m \text{ such that } |X_n(\omega) - X(\omega)| \geq \varepsilon\}) = 0$$

$$\iff \forall \varepsilon > 0, P(\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\} \text{ i.o.}) = 0$$

$$\iff \forall \varepsilon > 0, P(\limsup_{n \rightarrow \infty} \{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) = 0$$

$$\iff \forall \varepsilon > 0, P\left(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}\right) = 0$$

$$\iff \forall \varepsilon > 0, \lim_{m \rightarrow \infty} P\left(\bigcup_{n=m}^{\infty} \{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}\right) = 0.$$

As a consequence of this definition we get a version of the *Borel-Cantelli Lemma*:

As $n \rightarrow \infty$, $X_n \rightarrow X$ a.s. if $\forall \varepsilon > 0, \sum_{n=1}^{\infty} P(\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) < \infty$.

Convergence in Probability: (as in the **weak** law of large numbers)

As $n \rightarrow \infty$, $X_n \rightarrow X$ in probability or in measure

$$\iff \forall \varepsilon > 0, P(\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\iff \forall \varepsilon > 0, \lim_{m \rightarrow \infty} P(\{\omega : |X_m(\omega) - X(\omega)| \geq \varepsilon\}) = 0.$$

Since

$$P(\{\omega : |X_m(\omega) - X(\omega)| \geq \varepsilon\}) \leq P\left(\bigcup_{n=m}^{\infty} \{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}\right),$$

we see that convergence with probability one implies convergence in probability.

Convergence in Distribution: (as in the central limit theorem)

As $n \rightarrow \infty$, $X_n \rightarrow X$ in distribution or in law

$$\begin{aligned} \iff F_{X_n}(x) = P(\{\omega : X_n(\omega) \leq x\}) &\rightarrow P(\{\omega : X(\omega) \leq x\}) = F_X(x), \\ &\text{as } n \rightarrow \infty \forall x \text{ at which } F_X \text{ is continuous.} \end{aligned}$$

Suppose that each of the $\{X_n\}$ and X is a finite non-negative integer valued random variable, with

$$P(X_n = k) = p_k^{(n)} \text{ and } P(X = k) = p_k, \quad n = 1, 2, 3, \dots, \quad k = 0, 1, 2, \dots$$

Then $X_n \rightarrow X$ in distribution is equivalent to

$$\lim_{n \rightarrow \infty} p_k^{(n)} = p_k \text{ for } k = 0, 1, 2, \dots$$

Note: If F_X is continuous at x and $\varepsilon > 0$, then $X_n(\omega) \leq x$ implies either

$$\begin{aligned} X(\omega) &\leq x + \varepsilon \text{ or } |X_n(\omega) - X(\omega)| \geq \varepsilon \\ \implies F_{X_n}(x) &\leq F_X(x + \varepsilon) + P(\{\omega : |X_n(\omega) - X(\omega)| \geq \varepsilon\}) \\ \implies \limsup_{n \rightarrow \infty} F_{X_n}(x) &\leq F_X(x + \varepsilon) \text{ if } X_n \rightarrow X \text{ in probability.} \end{aligned}$$

Similarly,

$$\liminf_{n \rightarrow \infty} F_{X_n}(x) \geq F_X(x - \varepsilon) \text{ if } X_n \rightarrow X \text{ in probability.}$$

Since this is true $\forall \varepsilon > 0$ and F_X is continuous at x ,

$$\lim_{n \rightarrow \infty} F_{X_n}(x) = F_X(x).$$

Consequently, convergence in probability implies convergence in distribution.

One further note: convergence with probability one and convergence in probability require that all of the random variables involved are defined on the same probability space (since the probabilities involved in the definitions of these modes of convergence require joint distributions). This is not true for convergence in distribution since this mode of convergence only depends on the one-dimensional (marginal) distributions of these random variables.