Discrete Random Variables

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Fall, 2008

A *discrete* random variable X is a function on the sample space which takes either a finite number or a countably infinite number of real values and has a distribution described by its *probability mass function* or *pmf*

$$p(x) = P(\{X = x\}).$$

Note that $p(x) \geq 0$ and $\sum_x p(x) = 1$. In this case, if $I = (a, b]$ is an interval of real numbers, then

$$P(\{X \in I\}) = P(\{a < X \leq b\}) = \sum_{x \in I} p(x).$$

Furthermore, in this case we define the *mean* to be

$$\mu = E(X) = \sum_x xp(x)$$

provided that $\sum_x |x|p(x) < \infty$. If the mean μ exists, we define the *variance* to be

$$\sigma^2 = E([X - \mu]^2) = \sum_x (x - \mu)^2 p(x).$$

We then have the alternate calculation,

$$\sigma^2 = E(X^2) - \mu^2 = \sum_x x^2 p(x) - \mu^2.$$

Important Examples of Distributions of Discrete Random Variables:

1. The Discrete Uniform Distribution: We say that $X \sim \mathit{Uniform}\{x_1, x_2, \ldots, x_N\}$ or X is *equally likely* to take on each value in $\{x_1, x_2, \ldots, x_N\}$, if the *pmf* of X is given by

$$p(x) = \begin{cases} \frac{1}{N}, & \text{if } x \in \{x_1, x_2, \ldots, x_N\} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, this destribution is the basis for *classical* probability in which the probability of any event is the quotient of the number of outcomes belonging to that event and the number of outcomes, N, in the sample space of the experiment. It is easy to see that the mean for the this distribution is simply the average

$$\mu = \frac{1}{N}\sum_{j=1}^{N} x_j$$

and the variance is

$$\sigma^2 = \frac{1}{N}\sum_{j=1}^{N} x_j^2 - \mu^2.$$

2. The Bernoulli Distribution: We say that $X \sim \mathit{Bernoulli}(p)$ or that X is a *Bernoulli Trial* with probability of success p, where $0 \le p \le 1$, if the *pmf* of X is given by

$$p(x) = \begin{cases} p, \text{ if } x = 1 \\ q = 1 - p, \text{ if } x = 0 \\ 0, \text{ otherwise.} \end{cases}$$

   This distribution describes the result of an experiment in which there are only two categorical outcomes, “success” (which we code numerically by 1) and “failure” (which we code numerically by 0). The mean μ of this distribution is p and the variance $\sigma^2 = p(1-p) = pq$.

3. The Binomial Distribution: We say that $X \sim \mathit{Binomial}(n, p)$ or that X has a binomial distribution corresponding to n independent trials with common probability of success p on each trial, where n is a positive integer and $0 \le p \le 1$, if the *pmf* of X is given by

$$p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x}, \text{ if } x \in \{0, 1, \ldots, n\} \\ 0, \text{ otherwise.} \end{cases}$$

   The binomial distribution describes the number of successes in a sequence of n independent, identically distributed (i.i.d.)$\mathit{Bernoulli}(p)$ trials. A little direct work or some future knowledge about expected values shows that this distribution has mean $\mu = np$ and variance $\sigma^2 = npq = np(1-p)$.

4. The Geometric Distribution: We say that $X \sim \mathit{geometric}(p)$ or that X has a geometric distribution corresponding to a sequence of i.i.d.$\mathit{Bernoulli}(p)$ trials if the *pmf* of X is given by

$$p(x) = \begin{cases} pq^{x-1}, \text{ if } x \in \{1, 2, 3, \ldots\} \\ 0, \text{ otherwise.} \end{cases}$$

   This version of the geometric distribution represents the number of i.i.d. $\mathit{Bernoulli}(p)$ trials until the *first success*. Another version of this distribution represents the number of *failures* before the first success. For our version, calculations involving geometric series show that the mean $\mu = 1/p$ and the variance $\sigma^2 = q/p^2$.

5. The Poisson Distribution: We say that $x \sim \mathit{Poisson}(\lambda)$ or that X has a Poisson distribution with rate λ, where $\lambda > 0$, if the *pmf* of X is given by

$$p(x) = \begin{cases} e^{-\lambda}\lambda^x/x!, & \text{if } x \in \{0, 1, 2, \ldots\} \\ 0, & \text{otherwise.} \end{cases}$$

One way to obtain the Poisson distribution is to consider a sequence of binomial distributions in which the probability of success on each trial p_n depends upon the total number of trials n. If we assume that the expected number of successes, np_n, approaches a limit $\lambda > 0$ as $n \to \infty$ then the binomial probabilities converge to those of the Poisson distribution. Calculations involving the series for the exponential function yield the results that both the mean μ and the variance σ^2 for this distribution equal λ.

Review Exercises:

1. Show that for a discrete distribution with *pmf* p,

$$\sigma^2 = \sum_x x^2 p(x) - \mu^2.$$

2. If $X \sim \mathit{Binomial}(120, .4)$, find the mean μ and standard deviation σ for this random variable. Now calculate (by computer or by hand) the probabilities that $|X - \mu| \geq k\sigma$ for $k = 1, 2, 3$.

3. For $x = 0, 1, 2, 3, 4$, compare the probabilities that $X = x$ for the distributions: $\mathit{Binomial}(10, .2)$, $\mathit{Binomial}(100, .02)$, and $\mathit{Poisson}(2)$. What do you think?

4. Verify the formula for the mean of a geometric distribution.

5. Verify the formula for the mean of a Poisson distribution.

6. Derive the probability distribution for a random variable X which describes the number of trials until the *second* success in a sequence of i.i.d. $\mathit{Bernoulli}(p)$ trials.