Exponential Random Variables and Explosions

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Minimum Theorem: Let $\{W_k : k = 1, 2, \ldots, n\}$ be independent exponentially distributed random variables with $E(W_k) = 1/q_k$ for $k = 1, 2, \ldots, n$. Then $W = \min\{W_k : k = 1, 2, \ldots, n\}$ is exponentially distributed with

$$E(W) = 1/(q_1 + q_2 + \cdot + q_n).$$

Moreover, for any $k = 1, 2, \ldots, n$,

$$P(W_k = W) = q_k/(q_1 + q_2 + \cdot + q_n).$$

Proof: For any $t \geq 0$,

$$\begin{aligned} P(W > t) &= P(W_k > t, k = 1, 2, \ldots, n) \\ &= \prod_{k=1}^{n} P(W_k > t) \\ &= \prod_{k=1}^{n} e^{-q_k t} \\ &= e^{-(q_1+q_2+\cdot+q_n)t}. \end{aligned}$$

Furthermore,

$$\begin{aligned} P(W_k = W) &= \int_0^\infty P(W_k = W | W_k = w) q_k e^{-q_k w} dw \\ &= \int_0^\infty P(w = W | W_k = w) q_k e^{-q_k w} dw \\ &= \int_0^\infty P(W_j \geq w \text{ for } j \neq k | W_k = w) q_k e^{-q_k w} dw \\ &= \int_0^\infty P(W_j \geq w \text{ for } j \neq k) q_k e^{-q_k w} dw \\ &= \int_0^\infty \prod_{j \neq k} P(W_j \geq w) q_k e^{-q_k w} dw \\ &= \int_0^\infty \prod_{j \neq k} e^{-q_j w} q_k e^{-q_k w} dw \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty q_k e^{-(q_1+q_2+\cdot+q_n)w} dw \\
&= q_k/(q_1+q_2+\cdot+q_n).
\end{aligned}
$$

Convergence Theorem: Suppose that $\{W_n : n = 0, 1, 2, \ldots\}$ are independent non-negative random variables. Then

$$P(\sum_{n=0}^\infty W_n < \infty) > 0 \Longleftrightarrow \sum_{n=0}^\infty E(1-e^{-sW_n}) < \infty \text{ for some (every) } s > 0.$$

Proof: First suppose that $s > 0$ and

$$E(1-e^{-sW_n}) \to 0 \text{ as } n \to \infty$$

or, equivalently,

$$\ln\left[1 - E(1-e^{-sW_n})\right] \to 0 \text{ as } n \to \infty.$$

In this case, as $n \to \infty$,

$$
\begin{aligned}
-\ln\left[E(e^{-sW_n})\right] &= -\ln\left[1-E(1-e^{-sW_n})\right] \\
&\sim E(1-e^{-sW_n})
\end{aligned}
$$

in the sense that the ratio of corresponding terms in these sequences approach 1 as $n \to \infty$. Consequently, since

$$
\begin{aligned}
\sum_{n=0}^\infty -\ln\left[E(e^{-sW_n})\right] &= -\ln\left[\prod_{n=0}^\infty E\left(e^{-sW_n}\right)\right] \\
&= -\ln\left[E\left(\prod_{n=0}^\infty e^{-sW_n}\right)\right] \\
&= -\ln\left[E\left(e^{-s\sum_{n=0}^\infty W_n}\right)\right],
\end{aligned}
$$

the asymptotic comparison test for series with positive terms shows that,

$$\sum_{n=0}^\infty E(1-e^{-sW_n}) < \infty \Longleftrightarrow P(\sum_{n=0}^\infty W_n < \infty) > 0.$$

Now suppose that

$$P(\sum_{n=0}^\infty W_n < \infty) > 0.$$

Then, for every $s > 0$, we have

$$E\left(e^{-s\sum_{n=0}^\infty W_n}\right) > 0$$

or, equivalently,

$$\sum_{n=0}^{\infty} - \ln \left[E(e^{-sW_n}) \right] = \sum_{n=0}^{\infty} - \ln \left[1 - E(1 - e^{-sW_n}) \right] < \infty.$$

Since this implies that

$$\ln \left[1 - E(1 - e^{-sW_n}) \right] \to 0 \text{ as } n \to \infty,$$

we must have

$$\sum_{n=0}^{\infty} E(1 - e^{-sW_n}) < \infty.$$

Conversely, if for some $s > 0$ we assume that

$$\sum_{n=0}^{\infty} E(1 - e^{-sW_n}) < \infty,$$

then

$$E(1 - e^{-sW_n}) \to 0 \text{ as } n \to \infty.$$

Consequently,

$$P(\sum_{n=0}^{\infty} W_n < \infty) > 0.$$

Note 1: $L_n(s) = E(e^{-sW_n})$ is the Laplace transform of the distribution of W_n.

Note 2: Since the event $\{\sum_{n=0}^{\infty} W_n < \infty\}$ is a tail event, the only possible probabilities for this event are 0 and 1.

Note 3: Consequently, since

$$P(\sum_{n=0}^{\infty} W_n < \infty) > 0 \Longleftrightarrow P(\sum_{n=0}^{\infty} W_n = \infty) < 1,$$

the above theorem can be stated equivalently as

$$P(\sum_{n=0}^{\infty} W_n < \infty) = \begin{cases} 1, & \text{if } \sum_{n=0}^{\infty} E(1 - e^{-sW_n}) < \infty \\ 0, & \text{if } \sum_{n=0}^{\infty} E(1 - e^{-sW_n}) = \infty. \end{cases}$$

Corollary 1: If $\{W_n : n = 0, 1, 2, \ldots\}$ are independent exponentially distributed random variables with $E(W_n) = 1/q_n < \infty$ for $n = 0, 1, 2, \ldots$, then

$$P(\sum_{n=0}^{\infty} W_n = \infty) = 1 \Longleftrightarrow \sum_{n=0}^{\infty} E(W_n) = \sum_{n=0}^{\infty} 1/q_n = \infty.$$

Proof: If

$$P(\sum_{n=0}^{\infty} W_n = \infty) = 1,$$

then

$$\infty = \sum_{n=0}^{\infty} E(1-e^{-W_n}) \le \sum_{n=0}^{\infty} E(W_n).$$

Conversely, if

$$P(\sum_{n=0}^{\infty} W_n < \infty) > 0,$$

we have

$$\sum_{n=0}^{\infty} E(1-e^{-sW_n}) = \sum_{n=0}^{\infty} \left(1 - \frac{q_n}{s+q_n}\right) = \sum_{n=0}^{\infty} \frac{s}{s+q_n} < \infty \text{ for every } s > 0.$$

This shows that

$$\frac{s}{s+q_n} \to 0 \text{ as } n \to \infty \Longleftrightarrow q_n \to \infty \text{ as } n \to \infty.$$

Thus,

$$\frac{s}{s+q_n} \sim \frac{s}{q_n} \text{ as } n \to \infty$$

and, by the asymptotic comparison test for series with positive terms,

$$\sum_{n=0}^{\infty} \frac{s}{s+q_n} < \infty \Longleftrightarrow \sum_{n=0}^{\infty} 1/q_n < \infty.$$

Corollary 2: For a Markov pure jump process $\{X(t) : t \ge 0\}$ on state space $\mathcal{S}$ with embedded chain $\{X_n : n = 0, 1, 2, \ldots\}$ and jump rates $\{q_x : x \in \mathcal{S}\}$,

$$P_x(T_\infty < \infty) = P_x(\sum_{n=0}^{\infty} 1/q_{X_n} < \infty) \text{ for every } x \in \mathcal{S}.$$

Thus

$$\{X(t)\} \text{ is non} - \text{explosive } \Longleftrightarrow \text{ for every } x \in \mathcal{S},\ P_x(\sum_{n=0}^{\infty} 1/q_{X_n} = \infty) = 1.$$

Proof:

$$T_\infty = \sum_{n=0}^{\infty} (T_{n+1} - T_n) = \sum_{n=0}^{\infty} W(T_n) = \sum_{n=0}^{\infty} E_n/q_{X_n}.$$

Thus, by Corollary 1, since $\{W(T_n)\}$ are exponentially distributed and (conditionally) independent, given values of $\{X_n\}$, we see that

$$P(T_\infty < \infty | \{X_n\}) = \begin{cases} 1, \text{ if } \sum_{n=0}^{\infty} 1/q_{X_n} < \infty \\ 0, \text{ if } \sum_{n=0}^{\infty} 1/q_{X_n} = \infty \end{cases}$$

In other words,

$$P(T_\infty < \infty | \{X_n\}) = 1_{\left\{\sum_{n=0}^{\infty} 1/q_{X_n} < \infty\right\}}.$$

Taking expectations with respect to P_x, we get the stated result.

Corollary 3: The following conditions are *sufficient* for the Markov pure jump process $\{X(t) : t \geq 0\}$ to be non-explosive:

1. $\{q_x : x \in \mathcal{S}\}$ is bounded above, i.e. there is a real number $B > 0$ such that $q_x \leq B$ for all $x \in \mathcal{S}$.

2. $\mathcal{S}$ is a finite set.

3. If $\mathcal{T} \subset \mathcal{S}$ are the transient states of the embedded chain $\{X_n\}$, then

$$\text{for every } x \in \mathcal{S},\; P_x(X_n \in \mathcal{T} \text{ for every } n) = 0.$$

4. The embedded chain $\{X_n\}$ has no transient states.

Proof: If $q_x \leq B$ for all $x \in \mathcal{S}$, then $1/q_x \geq 1/B > 0$ for all $x \in \mathcal{S}$. Therefore,

$$\sum_{n=0}^{\infty} 1/q_{X_n} \geq \sum_{n=0}^{\infty} 1/B = \infty, \text{ for all values of } \{X_n\}.$$

If $\mathcal{S}$ is a finite set, we can take $B = \max\{q_x : x \in \mathcal{S}\}$. Next, if

$$P_x(X_n \in \mathcal{T} \text{ for every } n) = 0,$$

then with probability one there is a recurrent state, say $y(\omega)$, such that the chain $\{X_n(\omega)\}$ visits $y(\omega)$ infinitely often. Note that this state may have to be taken as different for different realizations of the process if there is more than one equivalence class of recurrent states. Let $\{n_j(\omega)\}$ be the values of n for which $X_n(\omega) = y(\omega)$. Therefore,

$$\sum_{n=0}^{\infty} 1/q_{X_n(\omega)} \geq \sum_{\{n: n=n_j(\omega)\}}^{\infty} 1/q_{y(\omega)} = \infty.$$

Finally, if $\{X_n\}$ has no transient states, then condition 3 must be satisfied.