More Properties of Generating Functions

H. Krieger, Mathematics 156, Harvey Mudd College

Fall, 2008

Theorem: Let N be a finite non-negative integer valued random variable so that $p_k = P(N = k)$ for $k = 0, 1, 2, \ldots$, and $\sum_{k=0}^{\infty} p_k = P(N < +\infty) = 1$. Let Φ be the generating function of N so that $\Phi(s) = E(s^N) = \sum_{k=0}^{\infty} s^k p_k$ with $\Phi(1) = 1$. If $0 \le p_1 < 1$ and $E(N) = \Phi'(1) \le 1$, then there is no solution of the equation $\Phi(s) = s$ in the interval $[0, 1)$. If $E(N) = \Phi'(1) > 1$ (which implies that $0 \le p_1 < 1$), then there is a unique solution of the equation $\Phi(s) = s$ in the interval $[0, 1)$.

Proof: Let $\Psi(s) = \Phi(s) - s$. Then

$$\Psi''(s) = \Phi''(s) = \sum_{k=2}^{\infty} k(k-1)s^{k-2}p_k \ge 0$$

so that Ψ is *convex* with $\Psi(1) = 0$. Moreover,

$$\Psi'(s) = \Phi'(s) - 1 = \sum_{k=1}^{\infty} ks^{k-1}p_k - 1 \le E(N) - 1$$

for $s \in [0, 1)$. Hence, if $0 \le p_1 < 1$ and $E(N) \le 1$, then $\Psi'(s) < 0$ for $s \in [0, 1)$ and the equation $\Psi(s) = 0$ has no solution in this interval. On the other hand, if $E(N) > 1$, then $\Psi'(1) > 0$ and $\Psi'(0) = p_1 - 1 < 0$ with $\Psi(0) = p_0 \ge 0$. Thus, there is a unique solution of the equation $\Psi(s) = 0$ in the interval $[0, 1)$.

Question: What happens if $p_1 = 1$?

Continuity Theorem: Suppose that $\{X_n : n = 1, 2, 3, \ldots\}$ are *finite* non-negative integer valued random variables so that $P(X_n = k) = p_k^{(n)}$, for $n = 1, 2, 3, \ldots,\ k = 0, 1, 2, \ldots$, with $\sum_{k=0}^{\infty} p_k^{(n)} = 1$, for $n = 1, 2, 3, \ldots$. Let Φ_n be the pgf for the random variable X_n. Then there exists a sequence $\{p_k\}$ such that

$$\lim_{n\to\infty} p_k^{(n)} = p_k \text{ for } k = 0, 1, 2, \ldots$$

if and only if there is a function $\Phi(s)$ defined for $0 < s < 1$ such that

$$\lim_{n\to\infty} \Phi_n(s) = \lim_{n\to\infty} \sum_{k=0}^{\infty} s^k p_k^{(n)} = \Phi(s) \text{ for } 0 < s < 1.$$

In this case, $\Phi(s) = \sum_{k=0}^{\infty} s^k p_k$. Moreover,

$$\sum_{k=0}^{\infty} p_k = 1 \text{ iff } \lim_{s\uparrow 1} \Phi(s) = 1.$$

Proof: First suppose that there exists a sequence $\{p_k\}$ such that

$$\lim_{n\to\infty} p_k^{(n)} = p_k \text{ for } k = 0, 1, 2, \ldots.$$

Note that in this case we have $0 \leq p_k \leq 1$ for $k = 0, 1, 2, \ldots$ and, in fact, $\sum_{k=0}^{\infty} p_k \leq 1$. Hence, $\Phi(s) = \sum_{k=0}^{\infty} s^k p_k$ is actually well defined for $0 \leq s \leq 1$. So, given $s \in (0, 1)$ and $\varepsilon > 0$, choose K so that $\sum_{k=K+1}^{\infty} s^k < \varepsilon/2$. Then observe that

$$|\Phi_n(s) - \Phi(s)| \leq \sum_{k=0}^{K} |p_k^{(n)} - p_k| + \sum_{k=K+1}^{\infty} s^k.$$

Therefore, if we choose M so that whenever $n \geq M$ we have

$$\sum_{k=0}^{K} |p_k^{(n)} - p_k| < \varepsilon/2,$$

we see that if $n \geq M$ then $|\Phi_n(s) - \Phi(s)| < \varepsilon$.

For the converse, assume that there is a function $\Phi(s)$ defined for $0 < s < 1$ such that

$$\lim_{n\to\infty} \Phi_n(s) = \lim_{n\to\infty} \sum_{k=0}^{\infty} s^k p_k^{(n)} = \Phi(s) \text{ for } 0 < s < 1.$$

Suppose that $\{p_k^{(n')}\}$ is a convergent subsequence of $\{p_k^{(n)}\}$, *i.e.*

$$\lim_{n'\to\infty} p_k^{(n')} = p_k$$

exists $\forall k$. Then,

$$\lim_{n'\to\infty} \Phi_{n'}(s) = \Phi(s)$$

so that Φ is the generating function of $\{p_k\}$. Consequently, *every* convergent subsequence of $\{p_k^{(n)}\}$ must have the same limit, namely $\{p_k\}$. By using a diagonal argument one can show that, in fact, every subsequence of $\{p_k^{(n)}\}$ has a further subsequence that does converge. Therefore, the original sequence $\{p_k^{(n)}\}$ must be convergent, with $\lim_{n\to\infty} p_k^{(n)} = p_k$ for $k = 0, 1, 2, \ldots$.

The result

$$\sum_{k=0}^{\infty} p_k = 1 \text{ iff } \lim_{s\uparrow 1} \Phi(s) = 1$$

is a direct consequence of the fact that

$$\lim_{s\uparrow 1} \Phi(s) = \sum_{k=0}^{\infty} p_k.$$