Converses to the Strong Law of Large Numbers

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Let $\{X_n\}$ be a sequence of i.i.d. random variables. The strong law of large numbers asserts that if

$$E\left(|X_1|\right) < \infty,$$

then as $n \to \infty$ the arithmetic means

$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$

converge to $E(X_1)$ with probability one.

In order to consider converses to this theorem, we need some results from probability theory. Let $\{X_n\}$ be any sequence of random variables. An event E in the σ-algebra generated by the sequence is called a *tail event* if and only if E belongs to the σ-algebra generated by the random variables $\{X_n, X_{n+1}, X_{n+2}, \ldots\}$ for every n. For example,

$$\left\{\left\{\frac{1}{n}\sum_{k=1}^{n} X_k\right\} \text{ converges}\right\}$$

and

$$\left\{\sum_{n=1}^{\infty} X_n \text{ converges}\right\}$$

are tail events. Similarly, if

$$X = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} X_k$$

exists with probability one, then $\{X < 0\}$ is a tail event, but if $\sum_{n=1}^{\infty} X_n$ converges with probability one, then

$$\left\{\sum_{n=1}^{\infty} X_n < 0\right\}$$

is not a tail event.

In the special case that the $\{X_n\}$ are *independent* random variables, the tail events have a simple structure which is described by *Kolmogorov's Zero-One Law:* In this case if E is a tail event, then $P(E)$ is either zero or one. The proof consists of showing that every tail event E is independent of itself so that

$$P(E) = P(E \cap E) = P(E)^2.$$

General Converse: Let $\{X_n\}$ be a sequence of i.i.d. random variables. Suppose that as $n \to \infty$ the averages

$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$

converge with positive probability. Then

$$E(|X_1|) < \infty.$$

Proof: Let $S_n = X_1 + X_2 + \cdots + X_n$. Since S_n/n converges with positive probability and the set of convergence is a tail event, we must actually have convergence with probability one. Moreover,

$$\frac{X_n}{n} = \frac{S_n}{n} - \frac{(n-1)}{n}\frac{S_{n-1}}{n-1}$$

so that X_n/n converges to 0 with probability one. Consequently,

$$P\left(\left\{\left|\frac{X_n}{n}\right| \geq 1\right\} \; i.o.\right) = 0$$

and hence, by the converse Borel-Cantelli lemma,

$$\begin{aligned} \sum_{n=1}^{\infty} P\left(\left\{\left|\frac{X_n}{n}\right| \geq 1\right\}\right) &= \sum_{n=1}^{\infty} P\left(\left\{\left|\frac{X_1}{n}\right| \geq 1\right\}\right) \\ &= \sum_{n=1}^{\infty} P\left(\{|X_1| \geq n\}\right) \\ &< \infty. \end{aligned}$$

However,

$$E(|X_1|) \leq \sum_{n=1}^{\infty} nP\left(\{n-1 \leq |X_1| < n\}\right) = 1 + \sum_{n=1}^{\infty} P\left(\{|X_1| \geq n\}\right).$$

Note that if $E(|X_1|) = \infty$, then we see that $\{S_n/n\}$ cannot converge on a set of positive probability. In other words,

$$P\left(\left\{\liminf_{n\to\infty} \frac{S_n}{n} < \limsup_{n\to\infty} \frac{S_n}{n}\right\}\right) = 1.$$

Special Converse: Suppose that $\{X_n\}$ is a sequence of i.i.d. *non-negative* random variables and $E(X_1) = \infty$. Then as $n \to \infty$ the averages

$$\frac{X_1 + X_2 + \cdots + X_n}{n}$$

diverge to ∞ with probability one.

Proof: Let $\alpha > 0$ and define the truncated random variables

$$X_n^\alpha = \begin{cases} X_n \text{ if } X_n < \alpha \\ \alpha \text{ if } X_n \geq \alpha. \end{cases}$$

Then $0 \leq X_n^\alpha \leq X_n$ and hence, with probability one,

$$\liminf_{n\to\infty} \frac{X_1 + X_2 + \cdots + X_n}{n} \geq \lim_{n\to\infty} \frac{X_1^\alpha + X_2^\alpha + \cdots + X_n^\alpha}{n} = E(X_1^\alpha).$$

Letting $\alpha \uparrow \infty$ gives the result since $E(X_1^\alpha) \uparrow E(X_1)$.