Homework Assignment #2

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Instructions: This assignment is due in class on Monday, September 15th. You are encouraged to work together on the problems but the final write-up that you submit must be done individually. You will need to read the handout on Non-negative Integer Valued Random Variables in order to solve these problems. If you are not comfortable with conditional expectations, then reading the handout on Conditional Probabilities and Expectations as Random Variables will be helpful.

1. Let N be a non-negative integer valued random variable which represents the number of failures before the first success in a sequence of independent Bernoulli trials with common probability of success p, where $0 < p < 1$. Find the generating function $\Phi(s) = E(s^N)$ of the distribution of N (which is one version of the geometric distribution).

2. Let $\{X_n : n = 1, 2, 3, ...\}$ be i.i.d. non-negative integer valued random variables independent of the non-negative integer valued random variable N and assume that

   $$E(X_1) < \infty, Var(X_1) < \infty, E(N) < \infty, \text{ and } Var(N) < \infty.$$

   Let S_N be the random sum

   $$\sum_{k=1}^{N} X_k.$$

   Use generating functions to show that

   $$Var(S_N) = E(N)Var(X_1) + [E(X_1)]^2 Var(N).$$

3. Let N and M be jointly distributed non-negative integer valued random variables and define, for $|s| \leq 1$, $|t| \leq 1$, the *joint* generating function

   $$\Phi_{N,M}(s,t) = E(s^N t^M) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} s^j t^k P(N = j, M = k).$$

Show that N and M are independent if and only if

$$\Phi_{N,M}(s,t) = \Phi_N(s)\Phi_M(t) \text{ forall } s \text{ and } t.$$

Hint: for the "if" part, which is the more difficult direction, show how the partial derivatives of $\Phi_{N,M}(s,t)$ at $(0,0)$ determine the joint distribution of N and M, i.e. the probabilities $P(N=j, M=k)$.

4. Suppose $\{X_n : n = 1,2,3,...\}$ are i.i.d. with common distribution given by

$$P(X_1 = k-1) = p_k \text{ for } k = 0,1,2,...\text{with } \sum_{k=0}^{\infty} p_k = 1.$$

Let $S_0 = X_0 = 1$ and for $n \geq 1$ define $S_n = \sum_{j=0}^{n} X_j$. The random variables $\{S_n : n = 0,1,2,...\}$ define a random walk on the integers, starting at 1, in which it is possible to jump any number of units in the positive direction but only one unit at a time in the negative direction (so that integers cannot be skipped when moving in the negative direction). Define the stopping time $N = \inf\{n : S_n = 0\}$, so that N represents the number of jumps until the origin is reached for the first time.

(a) Show that the generating function of N satisfies the equation

$$\Phi_N(s) = s\Phi\left(\Phi_N(s)\right),$$

where Φ is the generating function of the sequence $\{p_k\}$. Hint: first write

$$\Phi_N(s) = E(s^N) = \sum_{k=0}^{\infty} E(s^N | X_1 = k-1)P(X_1 = k-1)$$

and consider two cases. If $k = 0$, which means $X_1 = -1$, then $S_1 = X_0 + X_1 = 1 - 1 = 0$ and hence $N = 1$, so that $E(s^N | X_1 = -1) = s^1 = s$. On the other hand, if $k \geq 1$ which means $X_1 = k - 1 \geq 0$, then $S_1 = X_0 + X_1 = k$. Consequently, the distribution of N, given that $X_1 = k-1 \geq 0$, is the same as that of $1 + N_1 + N_2 + \cdots + N_k$, where the N_j's are independent and have the same distribution as N (since we have already used one jump to reach k and now we have to move k units in the negative direction to get to 0, one unit at a time).

(b) Apply the results of part (a) and of question #1 to find Φ_N in the case that $p_k = pq^k$ for $k = 0,1,2,...$, where $0 < p < 1$ and $q = 1 - p$. In particular, find $\Phi_N(1) = P(N < \infty)$ in terms of p. Hint: in solving for Φ_N, note that $\Phi_N(0) = P(N=0) = 0$ and for $P(N < \infty)$ distinguish between $p < 1/2$ and $p \geq 1/2$.