Homework Assignment #3

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Instructions: This assignment is due in class on Monday, September 22nd. You are encouraged to work together on the problems but the final write-up that you submit must be done individually.

1. Using your favorite computer program, e.g. Maple, Mathematica, Matlab, etc., provide a sample of 200 independent values of sample paths from time 0 through 100 of the Markov chain $\{X_n : n = 0, 1, 2, ...\}$ given by

$$X_0 = 0, \; X_n = h(X_{n-1}, U_n) \text{ for } n = 1, 2, 3, ...$$

   where the U_n are independent, identically $U(0, 1)$ distributed and

$$h(x, u) = \begin{cases} x/2 - 1, & \text{if } u \in (0, 1/2) \\ x/2 + 1, & \text{if } u \in [1/2, 1) \end{cases} .$$

2. Provide a sample of 200 independent values of sample paths from time 0 through 100 of the two-state Markov chains $\{X_n : n = 0, 1, 2, ...\}$ with state space $\{0, 1\}$, initial distribution $(.5, .5)$, and

   (a) transition probability matrix

$$\begin{bmatrix} .5 & .5 \\ .1 & .9 \end{bmatrix};$$

   (b) transition probability matrix

$$\begin{bmatrix} .1 & .9 \\ .5 & .5 \end{bmatrix}.$$

   Hint: the easiest way to generate the sample paths is by iterating functions (which you can find from the transition probabilities) $h_u \equiv h(\cdot, u) : \{0, 1\} \to \{0, 1\}$ selected randomly and independently by means of independent, identically $U(0, 1)$ distributed random variables so that $X_n = h(X_{n-1}, U_n)$, $n = 1, 2, 3, ...$

(c) For each of these Markov chains, find the distribution of X_{100} both approximately, via a histogram of the values of X_{100} that you generated, and by matrix multiplication.

(d) Explain why $\{X_n\}$ converges in distribution to a random variable X_∞ and give the distribution of X_∞.

3. In problem #2 you generated two-state Markov chains by letting

$$X_1 = h_{U_1}(X_0),\ X_2 = h_{U_2}(X_1) = h_{U_2}\left(h_{U_1}(X_0)\right) = \left(h_{U_2} \circ h_{U_1}\right)(X_0),$$

and, in general, for $n = 1, 2, 3, ...$ letting

$$X_n = \left(h_{U_n} \circ h_{U_{n-1}} \circ \cdots \circ h_{U_1}\right)(X_0).$$

Thus we can call $\{X_n : n = 0, 1, 2, ...\}$ the *forward* process for this Markov chain. On the other hand, we can define a new process $\{Y_n : n = 0, 1, 2, ...\}$, called the *backward* process for the Markov chain, by letting $Y_0 = X_0$ and for $n = 1, 2, 3, ...$

$$Y_n = \left(h_{U_1} \circ h_{U_2} \circ \cdots \circ h_{Un}\right)(X_0).$$

(a) Explain why $\{Y_n\}$ converges with probability one to a random variable Y_∞ and give the distribution of Y_∞ for the cases you considered in 2(a) and 2(b) above.

(b) How is this distribution of Y_∞ related to that of X_∞ above?

4. On the handout, Exercises on Distribution and Percentile Functions, answer the following questions:

(a) Exercise 1.
(b) Exercise 6.
(c) Exercise 10(a).
(d) Exercise 10(b).

5. In your text, *Introduction to Stochastic Processes*, by Hoel, Port, and Stone, read Sections 1.1-1.3 and solve the following exercises in Chapter 1:

(a) Exercise 2.
(b) Exercise 6.