How to Recognize a Markov Chain on a State Space $\mathcal{S}$

H. Krieger, Mathematics 156, Harvey Mudd College

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Theorem: Let $\xi_0, \xi_1, \xi_2, \ldots$ be *independent* random elements of a set E, with $\xi_1, \xi_2, \xi_3, \ldots$ *identically distributed*. Let $g : E \to \mathcal{S}$ and $h : \mathcal{S} \times E \to \mathcal{S}$. Define $X_0 = g(\xi_0)$ and, for $n \geq 0$, define $X_{n+1} = h(X_n, \xi_{n+1})$. Then $\{X_n : n \geq 0\}$ is a Markov Chain on $\mathcal{S}$.

Proof: First note that, by induction, X_n is a function of the random elements $\xi_0, \xi_1, \ldots, \xi_n$. Since ξ_{n+1} is independent of $\xi_0, \xi_1, \ldots, \xi_n$, we see that ξ_{n+1} is independent of $X_0, X_{1,\ldots,}X_n$. Consequently, we can verify the Markov property (for a discrete state space) as follows:

$P(X_{n+1} = x_{n+1} \,|X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0\,)$
$= P(h(X_n, \xi_{n+1}) = x_{n+1} \,|X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0\,)$
$= P(h(x_n, \xi_{n+1}) = x_{n+1} \,|X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_0 = x_0\,)$
$= P(h(x_n, \xi_{n+1}) = x_{n+1})$
$= P(h(x_n, \xi_{n+1}) = x_{n+1} \,|X_n = x_n\,)$
$= P(h(X_n, \xi_{n+1}) = x_{n+1} \,|X_n = x_n\,)$
$= P(X_{n+1} = x_{n+1} \,|X_n = x_n\,)$.

The proof for a general state space is similar.

An interpretation of this theorem is that, for each $\xi \in E$, $h_\xi = h(\cdot, \xi)$ is a mapping from $\mathcal{S}$ into $\mathcal{S}$ and the Markov Chain proceeds by iterating the *independent identically distributed* maps $h_{\xi_1}, h_{\xi_2}, h_{\xi_3}, \ldots$. In other words,

$$\begin{aligned}
X_1 &= h(X_0, \xi_1) = h_{\xi_1}(X_0), \\
X_2 &= h(X_1, \xi_2) = h_{\xi_2}(X_1) = (h_{\xi_2} \circ h_{\xi_1})(X_0), \\
X_3 &= h(X_2, \xi_3) = h_{\xi_3}(X_2) = (h_{\xi_3} \circ h_{\xi_2} \circ h_{\xi_1})(X_0), \\
& \text{etc.}
\end{aligned}$$

If we refer to the process $\{X_n : n \geq 1\}$ as the *forward* process generated by the maps $\{h_{\xi_n} : n \geq 1\}$, we can also construct a *backward* process $\{Y_n : n \geq 1\}$ generated by the maps $\{h_{\xi_n} : n \geq 1\}$ by letting

$$Y_n = (h_{\xi_1} \circ h_{\xi_2} \circ \cdots \circ h_{\xi_n})(X_0).$$