Verifying the Markov Property in Continuous Time

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Review: Let $\{X_n : n = 0, 1, 2, \ldots\}$ be a discrete time Markov chain on the finite or countably infinite state space $\mathcal{S}$ with transition probability matrix Q such that $Q(x, x) = 0$ for all $x \in \mathcal{S}$. For each $x \in \mathcal{S}$, let $q_x \geq 0$ be the holding time parameter in the state x (*i.e.* the probability of remaining in state x for an amount of time greater than t is $e^{-q_x t}$, so that $1/q_x$ is the mean of the exponentially distributed holding time in state x if $q_x > 0$ and $q_x = 0$ whenever x is an absorbing state.) Let $\{E_n : n = 0, 1, 2, \ldots\}$ be a sequence of independent, identically distributed exponential random variables, each with mean 1, and independent of the $\{X_n\}$.

Definitions: Let $T_0 = 0$ and for $n \geq 1$ let

$$T_n = \sum_{k=0}^{n-1} \frac{E_k}{q_{X_k}}.$$

We then define, for $0 \leq t < T_\infty = \lim_{n\to\infty} T_n$,

$$X(t) = \sum_{n=0}^{\infty} X_n 1_{[T_n, T_{n+1})}(t).$$

Finally, for x and y in $\mathcal{S}$ and $0 \leq t < T_\infty$, we define the transition probability function

$$P_{x,y}(t) = P(X(t) = y | X(0) = x) = P_x(X(t) = y).$$

Theorem: If x, y_1, and y_2 belong to $\mathcal{S}$ and $0 < t_1 < t_2 < T_\infty$, then

$$P_x(X(t_1) = y_1, X(t_2) = y_2) = P_{x,y_1}(t_1) P_{y_1,y_2}(t_2 - t_1).$$

Proof: We first note that we can write

$$P_{x,y}(t) = \sum_{n=0}^{\infty} P_x(X_n = y, T_n \leq t < T_{n+1}).$$

Now let's examine the terms in this sum. For $n = 0$,

$$P_x(X_0 = y, T_0 \leq t < T_1) = \delta_{x,y} e^{-q_x t}.$$

On the other hand, if $n \geq 1$,

$$P_x(X_n = y, T_n \leq t < T_{n+1})$$

is the sum over $x_1, \ldots, x_{n-1} \in \mathcal{S}$ of

$$P_x(X_j = x_j, 1 \leq j < n, X_n = y, T_n \leq t < T_{n+1}),$$

where

$$P_x(X_j = x_j, 1 \leq j < n, X_n = y, T_n \leq t < T_{n+1})$$

is the product of

$$P_x(T_n \leq t < T_{n+1} | X_j = x_j, 1 \leq j < n, X_n = y)$$

and

$$P_x(X_j = x_j, 1 \leq j < n, X_n = y).$$

Thus if $F = F_{x,x_1,\ldots,x_{n-1}}$ is the distribution function of

$$E_0/q_x + E_1/q_{x_1} + \cdots + E_{n-1}/q_{x_{n-1}},$$

i.e. the distribution function of T_n given $X_0 = x, X_1 = x_1, , X_{n-1} = x_{n-1}$, we see that

$$\begin{aligned}
& P_x(T_n \leq t < T_{n+1} | X_j = x_j, 1 \leq j < n, X_n = y) \\
= & \int_0^t P_x(T_n \leq t < T_{n+1} | X_j = x_j, 1 \leq j < n, X_n = y, T_n = s) dF(s) \\
= & \int_0^t P(E_n/q_y > t - s) dF(s) \\
= & \int_0^t \int_{t-s}^{\infty} q_y e^{-q_y u} du \; dF(s) \\
= & \int_0^t e^{-q_y(t-s)} \; dF_{x,x_1,\ldots,x_{n-1}}(s).
\end{aligned}$$

Note that the expression for $n = 0$ can be interpreted as a special case of the above if we let the distribution function F be concentrated at 0. Moving to the case of two time points, we can write

$$P_x(X(t_1) = y_1, X(t_2) = y_2) =$$

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} P_x(X_n = y_1, T_n \leq t_1 < T_{n+1}, X_{n+m} = y_2, T_{n+m} \leq t_2 < T_{n+m+1}).$$

Thus once again we see that

$$P_x(X_n = y_1, T_n \leq t_1 < T_{n+1}, X_{n+m} = y_2, T_{n+m} \leq t_2 < T_{n+m+1})$$

can be written as the sum over $x_1, \ldots, x_{n-1}, \ldots, x_{n+1}, \ldots, x_{n+m-1} \in \mathcal{S}$ of

$P_x(X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2, T_n \le t_1 < T_{n+1}, T_{n+m} \le t_2 < T_{n+m+1})$,

where

$P_x(X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2, T_n \le t_1 < T_{n+1}, T_{n+m} \le t_2 < T_{n+m+1})$

is the product of

$P_x(T_n \le t_1 < T_{n+1}, T_{n+m} \le t_2 < T_{n+m+1} | X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2)$

and

$$P_x(X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m-1} = x_{n+m-1}, X_{n+m} = y_2).$$

Using the Markov property for the chain $\{X_n\}$, we see that

$$P_x(X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m-1} = x_{n+m-1}, X_{n+m} = y_2)$$

is the product of

$$P_x(X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = y_1)$$

and

$$P_{y_1}(X_1 = x_{n+1}, \ldots, X_{m-1} = x_{n+m-1}, X_m = y_2).$$

Moreover, again letting $F = F_{x,x_1,\ldots,x_{n-1}}$ be the conditional distribution of T_n,

$P_x(T_n \le t_1 < T_{n+1}, T_{n+m} \le t_2 < T_{n+m+1} | X_1 = x_1, \ldots, X_n = y_1, \ldots, X_{n+m} = y_2)$

can be written as

$$\int_0^{t_1} \int_{t_1 - s}^{\infty} q_{y_1} e^{-q_{y_1} u} P_x \left(s + u + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \le t_2 < s + u + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} + \frac{E_{n+m}}{q_{y_2}} \right) du \, dF(s).$$

If we make the change of variable, $v = u - (t_1 - s) = s + u - t_1$, we get

$$\int_0^{t_1} \int_0^{\infty} q_{y_1} e^{-q_{y_1}(v + t_1 - s)} P \left(v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \le t_2 - t_1 < v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} + \frac{E_{n+m}}{q_{y_2}} \right) dv \, dF(s),$$

which is the product of two integrals,

$$\int_0^{t_1} e^{-q_{y_1}(t_1 - s)} dF(s)$$

and

$$\int_0^{\infty} q_{y_1} e^{-q_{y_1} v} P \left(v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} \le t_2 - t_1 < v + \sum_{k=n+1}^{n+m-1} \frac{E_k}{q_{x_k}} + \frac{E_{n+m}}{q_{y_2}} \right) dv.$$

However, the first of these integrals is just

$$P_x(T_n \le t_1 < T_{n+1} | X_1 = x_1, \ldots, X_{n-1} = x_{n-1}, X_n = y_1)$$

and the second is

$$P_{y_1}(T_m \le t_2 - t_1 < T_{m+1} | X_1 = x_{n+1}, \ldots, X_{m-1} = x_{n+m-1}, X_m = y_2).$$

Consequently, putting these pieces together (by multiplying the conditional probabilities by the events we conditioned on and then summing these results), we get the desired result

$$P_x(X(t_1) = y_1, X(t_2) = y_2) = P_{x,y_1}(t_1) P_{y_1,y_2}(t_2 - t_1).$$