Martingales

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Definitions: A sequence $\{X_n : n = 0, 1, 2, \ldots\}$ of random variables, with finite expected values and defined on the same probability space, is called a *martingale* if and only if

$$E(X_{n+1}|X_n, X_{n-1}, \ldots, X_0) = X_n$$

for $n = 0, 1, 2, \ldots$. Similarly, such a sequence is called a *sub-martingale* if and only if

$$E(X_{n+1}|X_n, X_{n-1}, \ldots, X_0) \geq X_n$$

for $n = 0, 1, 2, \ldots$ and is called a *super-martingale* if and only if

$$E(X_{n+1}|X_n, X_{n-1}, \ldots, X_0) \leq X_n$$

for $n = 0, 1, 2, \ldots$.

If we think of the random variable X_0, often a given constant, as representing the initial fortune of a gambler and X_n for $n \geq 1$ as representing his/her fortune after n gambles, then a martingale is said to represent a gambling game which is "fair" in the sense that the conditional expectation of the gambler's fortune after a gamble, given the past history of the game, is the value of the fortune before before that gamble. So, in what sense are sub-martingales and super-martingales "sub-fair" and "super-fair"? Certainly, the way these games are defined, the "sub-fair" and "super-fair" descriptions do not apply to the gambler's fortune since the conditional expectation of that fortune is increasing in the first case and decreasing in the second case. On the other hand, if we think of the game from the point of view of the casino, the adjectives are consistent. Thinking in terms of the casino is the way to remember which inequality goes with which definition.

One important property of Martingales is that $E(X_n) = E(X_0)$ for all $n \geq 1$. This follows by induction from the fact that

$$E(X_{n+1}) = E(E(X_{n+1}|X_n, X_{n-1}, \ldots, X_0)) = E(X_n)$$

for $n \geq 0$. Similarly, for sub-martingales we have $E(X_{n+1}) \geq E(X_n)$ for $n \geq 0$ and for super-martingales we have $E(X_{n+1}) \leq E(X_n)$.

Note that if $\{X_n\}$ is a Markov chain on a state space $\mathcal{S} \subset \mathbb{R}$, with transition probability matrix P, then

$$E(X_{n+1}|X_n, X_{n-1}, \ldots, X_0) = E(X_{n+1}|X_n)$$

and

$$E(X_{n+1}|X_n = x) = \sum_{y\in\mathcal{S}} P(x,y)y$$

for every $x \in \mathcal{S}$. Consequently, the definitions for martingale, sub-martingale, and super-martingale reduce, respectively, to

$$\sum_{y\in\mathcal{S}} P(x,y)y = x \text{ for } x \in \mathcal{S},$$

$$\sum_{y\in\mathcal{S}} P(x,y)y \geq x \text{ for } x \in \mathcal{S}, \text{ and}$$

$$\sum_{y\in\mathcal{S}} P(x,y)y \leq x \text{ for } x \in \mathcal{S}.$$

Martingale transforms: Let $\{X_n\}$ be a sequence of random variables that is either a martingale, a sub-martingale, or a super-martingale. Consider a sequence of random variables $\{C_n\}$ constructed as follows: C_0 is a function of X_0; C_1 is a function of C_0, X_0, and X_1 (and hence, ultimately, of X_0 and X_1); C_2 is a function of C_0, C_1, X_0, X_1, and X_2 (and hence, ultimately, of X_0, X_1, and X_2; etc. Such a sequence $\{C_n\}$ is said to be "adapted" to the sequence $\{X_n\}$.

Define the martingale transform of $\{X_n\}$ by $\{C_n\}$ to be the sequence $\{Y_n\}$ given by $Y_0 = 0$ and for $n \geq 1$

$$Y_n = \sum_{k=1}^{n} C_{k-1}[X_k - X_{k-1}].$$

Here's the gambling interpretation of the martingale transform: Suppose $\{X_n\}$ represents the gamblers fortune when each bet is made with a fixed unit stake. For $n \geq 0$, let C_n represent the amount bet on the $(n+1)$st game, where this decision can be made only on the basis of what has happened through the end of the nth game. Then Y_n represents the change in fortune for the gambler from its initial value through the end of the nth game using this strategy. The martingale transform is the discrete time analog of the stochastic integral for continuous time.

Theorem (You can't beat the system!): Let $\{X_n\}$ be a martingale. Suppose that $\{C_n\}$ is bounded and adapted to $\{X_n\}$. Let $\{Y_n\}$ be the martingale transform of $\{X_n\}$ with respect to $\{C_n\}$. Then $\{Y_n\}$ is a martingale.

Proof: First note that, for $n \geq 0$, $Y_{n+1} = C_n[X_{n+1} - X_n] + Y_n$. Therefore,

$$E(Y_{n+1}|Y_n, \ldots, Y_0) = E(C_n[X_{n+1} - X_n]|Y_n, \ldots, Y_0) + Y_n.$$

On the other hand,

$$E(C_n[X_{n+1} - X_n]|Y_n, \ldots, Y_0) = E\left(E(C_n[X_{n+1} - X_n]|X_n, \ldots, X_0)|Y_n, \ldots, Y_0\right)$$

and, since $\{X_n\}$ is a martingale,

$$E(C_n[X_{n+1} - X_n]|X_n, \ldots, X_0) = C_n[E(X_{n+1}|X_n \ldots, X_0) - X_n] = C_n[0] = 0.$$

Corollary 1 (You can't change the system!): Let $\{X_n\}$ be a sub-martingale (super-martingale). Suppose that $\{C_n\}$ is *non-negative*, bounded, and adapted to $\{X_n\}$. Let $\{Y_n\}$ be the martingale transform of $\{X_n\}$ with respect to $\{C_n\}$. Then $\{Y_n\}$ is a sub-martingale (super-martingale).

Proof: In the last step of the above proof, the non-negativity of C_n means that the sign of

$$C_n[E(X_{n+1}|X_n \ldots, X_0) - X_n]$$

is the same as that of

$$E(X_{n+1}|X_n \ldots, X_0) - X_n.$$

Important Special Case: Let τ be a stopping time with respect to $\{X_n\}$ and define $C_n = 1_{\{\tau > n\}}$ so that $C_n = 1$ for $n < \tau$ and $C_n = 0$ for $n \geq \tau$. In the gambling context, τ represents the number of bets made by the gambler before stopping his/her play. Therefore, in this context, $\{C_n\}$ represents a betting strategy that consists of betting a fixed amount on every game until a decision is made to stop. Then, the change in fortune $\{Y_n\}$ using this strategy is given for $n \geq 1$ by

$$\begin{aligned} Y_n &= \sum_{k=1}^{n} C_{k-1}[X_k - X_{k-1}] \\ &= \sum_{k=1}^{n} 1_{\{\tau \geq k\}}[X_k - X_{k-1}] \\ &= \sum_{k=1}^{n \wedge \tau}[X_k - X_{k-1}] \\ &= X_{n \wedge \tau} - X_0. \end{aligned}$$

Here, the random variable $X_{n\wedge\tau}$ equals X_n if $\tau \geq n$ but equals X_k in the case that $\tau = k < n$. Therefore, $Y_0 = 0$ is also consistent with this formula. Note that since $\{C_n\}$ is non-negative, bounded, and adapted to $\{X_n\}$, both the above theorem and its corollary apply.

Corollary 2: Suppose that $\{X_n\}$ is a martingale, $P(\{\tau < \infty\}) = 1$, and $\{X_{n\wedge\tau}\}$ is bounded. Then $E(X_\tau) = E(X_0)$.

Proof: We know that the random variables $Y_n = X_{n\wedge\tau} - X_0$ form a martingale and, hence, for every $n \geq 1$

$$E(Y_n) = E(X_{n\wedge\tau} - X_0) = E(Y_0) = 0.$$

Therefore,

$$\lim_{n\to\infty} E(X_{n\wedge\tau}) = E(X_0).$$

Since, $P(\{\tau < \infty\}) = 1$, $n \wedge \tau \to \tau$ with probability 1 as $n \to \infty$ and thus $X_{n\wedge\tau} \to X_\tau$ with probability 1 as $n \to \infty$. Invoking the bounded convergence theorem allows us to assert

$$E(X_0) = \lim_{n\to\infty} E(X_{n\wedge\tau}) = E(X_\tau).$$

For example, if $\{X_n\}$ is a symmetric random walk on the integers $\mathcal{S} = \{a, a+1, \ldots, b-1, b\}$ and $X_0 = x \in (a, b)$, let τ be the duration of the walk. Then $\{X_n\}$ is a bounded martingale that is a Markov chain with state space $\mathcal{S}$. Furthermore, since this state space is finite with a and b the only recurrent states, $\tau = \tau_{\mathcal{S}_R}$, the time of absorption into the set of recurrent states $\mathcal{S}_R$, we have

$$\rho_{x,a} + \rho_{x,b} = \rho_{\mathcal{S}_R}(x) = P_x(\{\tau < \infty\}) = 1.$$

Also, since $\rho_{x,a} = P_x(\{X_\tau = a\})$ and $\rho_{x,b} = P_x(\{X_\tau = b\})$, we see that

$$E_x(X_\tau) = x \text{ is equivalent to } a\rho_{x,a} + b\rho_{x,b} = x.$$

In other words, $(b-a)\rho_{x,b} = x - a$, so that

$$\rho_{x,b} = \frac{x-a}{b-a} \text{ and } \rho_{x,a} = \frac{b-x}{b-a}.$$