Computing a Stationary Distribution

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Theorem: Let P be the transition matrix for an irreducible Markov chain $\{X_n\}$ with a finite number, r, of states. Let π be the unique stationary distribution, written as an r-dimensional row vector. Let I be the $r \times r$ identity matrix, let u be the r-dimensional uniform distribution, written as a row vector of all $1/r$'s, and let U be the $r \times r$ matrix with each row equal to u. Then π is the unique solution of the equation

$$\pi[I - P + U] = u,$$

namely,

$$\pi = u[I - P + U]^{-1}.$$

Proof: Since $\pi = \pi P$, *i.e.* $\pi(I - P) = 0$, and $\pi U = u$, we see that π satisfies the given equation. Consequently, we need to show that π is the *unique* solution of this equation and this is equivalent to showing that the matrix $I - P + U$ is non-singular (or invertible). One way to do this is to show that if v is an r-dimensional column vector and

$$[I - P + U]v = 0,$$

then $v = 0$. Now,

$$\begin{aligned} [I - P + U]v = 0 \quad &\Rightarrow \quad \pi\left([I - P + U]v\right) = 0 \\ &\Rightarrow \quad \left(\pi[I - P + U]\right)v = 0 \\ &\Rightarrow \quad u \cdot v = 0. \end{aligned}$$

In other words, the arithmetic mean of the components of v must be 0. Consequently, $Uv = 0$, so that $[I - P]v = 0$. Therefore, $v = Pv$ and hence, for all positive integers k, we have $v = P^k v$. This, in turn, implies that

$$v = \frac{1}{n}\sum_{k=1}^{n} P^k v.$$

Letting $n \to \infty$, we see that $v = \Pi v$, where Π is the $r \times r$ matrix with each row equal to π. Consequently, each component of v is identical and, since the arithmetic mean of the components is 0, we must have $v = 0$.

To get some insight into why this method works, let $Q = P - U$, so that $I - P + U = I - Q$. Note that

$$Q^2 = [P - U]P = P^2 - UP,$$

since $U^2 = PU = U$. By induction, we get, for $n \geq 2$,

$$Q^n = [P - U]P^{n-1} = P^n - UP^{n-1}.$$

Therefore,

$$u[I+Q+Q^2+\cdots+Q^n] = u+(uP-u)+(uP^2-uP)+\cdots+(uP^n-uP^{n-1}) = uP^n,$$

the row vector which represents the distribution of X_n when the initial distribution is uniform. Hence if $\{X_n\}$ is aperiodic, we know that

$$u[I + Q + Q^2 + \cdots + Q^n] = uP^n \to \pi = u[I - Q]^{-1}.$$

In particular, this argument shows that $u[I-Q]^{-1}$ must, in fact, be a probability distribution, something which is not obvious in general.