

Homework 6 - KEY

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Note: Many problems can be solved in more than one way; we present only a single solution here.

1 Problem 2-21abd

For part a, we obtain the cumulative distribution function by taking the integral of the probability density function with respect to x :

$$\int \frac{2x}{9} dx = \boxed{\frac{x^2}{9}}$$

on the interval $(0, 3)$. Below this interval, it is 0, and above, it is 1.

For part b, we simply need to perform the usual integrals. There are no special tricks to this.

$$E[X] = \int_0^3 x \cdot \frac{2x}{9} dx = \left(\frac{2x^3}{27}\right)_0^3 = \boxed{2}$$

$$E[X^2] = \int_0^3 x^2 \cdot \frac{2x}{9} dx = \left(\frac{x^4}{18}\right)_0^3 = 4.5$$

$$\text{Var}(X) = E[X^2] - E[X]^2 = 4.5 - 2^2 = \boxed{0.5}$$

For part d, we first note that the two intervals are disjoint, and their union is the entire real line, so their probabilities must sum to 1. Since both have equal probabilities, it follows that $P(X \leq m) = \frac{1}{2}$. Therefore, we must have:

$$\int_0^m \frac{2x}{9} dx = \left(\frac{x^2}{9}\right)_0^m = \frac{m^2}{9} = \frac{1}{2}$$

It follows that $\boxed{m = \frac{3\sqrt{2}}{2}}$.

2 Problem 6A

For part a, the cdf of X is again obtained by taking the integral of the pdf with respect to x ; thus, the cdf is

$$\int (90x^8 - 90x^9)dx = \boxed{10x^9 - 9x^{10}}$$

on the interval $(0, 1)$. Again, below this interval, it is 0, and above, it is 1. The graphs are not shown here, but can be easily produced with the aid of a graphing calculator or appropriate software.

For part b, we simply evaluate the cdf for $x = 0.5$. We find that

$$P(X \leq 0.5) = 10 \cdot (0.5)^9 - 9 \cdot (0.5)^{10} = \boxed{0.010742}$$

For part c, we first note that since this is a continuous distribution, the probability that X is equal to any given point is 0. Thus, $P(0.25 < X \leq 0.5) = P(0.25 \leq X \leq 0.5)$. To actually find the probability, we can simply find the probability that $P(X \leq 0.25)$ and subtract it from our result in the previous problem. Thus, the solution is:

$$P(X \leq 0.25) = .010742 - (10 \cdot (0.25)^9 - 9 \cdot (0.25)^{10}) = \boxed{0.010713}$$

For part d, no special tricks are necessary. We simply perform the necessary integrals. Note that since the pdf is only non-zero on $(0, 1)$, we need not integrate over all real numbers.

$$E[X] = \int_0^1 x \cdot (90x^8 - 90x^9)dx = (9x^{10} - \frac{90}{11}x^{11})_0^1 = \boxed{0.81818}$$

$$E[X^2] = \int_0^1 x^2 \cdot (90x^8 - 90x^9)dx = (\frac{90}{11}x^{11} - \frac{15}{2}x^{12})_0^1 = 0.68182$$

$$\sigma_X = \sqrt{E[X^2] - E[X]^2} = \sqrt{0.68182 - 0.81818^2} = \boxed{0.11134}$$

Finally, for part e, since we now have the mean and standard deviation, we find that the range of values in question is the interval $[0.706841, .929522]$. We again use the cumulative distribution function to find:

$$P(0.706841 \leq X \leq 0.929522) = 0.846584 - 0.160255 = \boxed{0.686329}$$

3 Problem 6B

To solve this problem, we merely need to take advantage of the linear nature of the mean and variance of a random variable:

$$E[F] = 1.8 \cdot E[C] + 32 = 1.8 \cdot 120 + 32 = \boxed{248}$$

$$\sigma_F = 1.8 \cdot \sigma_C = 3.24 \cdot 2 = \boxed{6.48}$$

4 Problem 6C

We wish to find the expected value for the area of the sampling region, which can be defined as πR^2 . Thus, we wish to find

$$E[\pi R^2] = \pi \cdot E[R^2]$$

From here, we simply find $E[R^2]$ as we would normally.

$$E[R^2] = \int_9^{11} r^2 \cdot \left(\frac{3}{4}[1 - (10 - r)^2]\right) dr = \left(\frac{-99}{4}r^3 + \frac{15}{4}r^4 - \frac{3}{20}r^5\right)_9^{11} = 100.2$$

$$E[\pi R^2] = \boxed{100.2\pi}$$

5 Problem 6D

For each part, we compute $E[X^n]$ as we would any other expected value. Note that in this case, we must integrate over the interval $[\theta, \infty]$.

For part a, the integral is facilitated by the fact that $k > 1$; otherwise, we would not know how x behaves upon integration.

$$\int_{\theta}^{\infty} x \cdot \frac{k \cdot \theta^k}{x^{k+1}} dx = k \cdot \theta^k \left(\frac{-1}{(k-1)x^{k-1}} \right)_{\theta}^{\infty} = \boxed{\frac{k\theta}{k-1}}$$

For part b, if we assume $k = 1$, then our integral becomes:

$$\int_{\theta}^{\infty} x \cdot \frac{\theta}{x^2} dx = \theta \cdot \int_{\theta}^{\infty} \frac{1}{x} dx = \theta \ln(x)_{\theta}^{\infty}$$

which is infinite. Thus, the mean is undefined.

For part c, the integral is again made possible thanks to the fact that $k > 2$.

$$E[X^2] = \int_{\theta}^{\infty} x^2 \cdot \frac{k \cdot \theta^k}{x^{k+1}} dx = k \cdot \theta^k \left(\frac{-1}{(k-2)x^{k-2}} \right)_{\theta}^{\infty} = \frac{k\theta^2}{k-2}$$

$$Var[X] = E[X^2] - E[X]^2 = \frac{k\theta^2}{k-2} - \left(\frac{k\theta}{k-1} \right)^2 = \boxed{\frac{k\theta^2}{(k-1)^2(k-2)}}$$

For part d, if we assume $k = 2$, then we find:

$$E[X^2] = \int_{\theta}^{\infty} x^2 \cdot \frac{2 \cdot \theta^2}{x^3} dx = 2 \cdot \theta^2 (\ln(x))_{\theta}^{\infty}$$

which, again, is infinite. Thus, the variance is undefined.

For part e, we find that in general,

$$E[X^n] = \int_{\theta}^{\infty} x^n \cdot \frac{k \cdot \theta^k}{x^{k+1}} dx = k \cdot \theta^k \int_{\theta}^{\infty} x^{n-k-1} dx$$

This only evaluates to a finite quantity when $-1 > n - k - 1$, that is, $\boxed{k > n}$.