Selected Topics in Fractional Graph Theory

by

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Abstract

We may formulate the chromatic number $\chi(G)$ of a graph G as follows: choose as few independent sets as possible such that every vertex of G is in at least one of the chosen sets. This is easily written as a $\{0,1\}$-integer program. We define the *fractional chromatic number* $\chi_f(G)$ to be the value of the linear relaxation of this program. The integer dual of this program yields the clique number $\omega(G)$ of the graph, and we define *fractional clique number* $\omega_f(G)$ to be the value of the linear relaxation of the integer dual. By strong linear programming duality, $\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G)$ for any finite graph.

We may equivalently write $\chi_f(G) = \lim_{b\to\infty} \chi_b(G)/b$, where $\chi_b(G)$ is the fewest total colors need in order to assign each vertex a set of b distinct colors disjoint from the color sets of its neighbors.

In Chapter 2, for infinite graph G, we define $\overline{\chi_f}(G)$ to be the supremum of χ_f of all G's finite subgraphs. We answer an open problem of Leader [8] by constructing infinite graphs for which $\overline{\chi_f} < \chi_f < \infty$. Further, by showing that $\overline{\chi_f} = \omega_f$, we disprove the result of strong duality for infinite graphs.

For Chapter 3, we define the *fractional Ramsey number* $r_f(x,y)$ simply by replacing ω with ω_f in the definition of ordinary Ramsey numbers. We derive an exact formula for $r_f(x,y)$, which is on the order of xy. We also consider a multi-color version and several relaxations of fractional Ramsey number.

In Chapter 4, we consider the *fractional dimension* of posets of trees. Fractional dimension comes from the linear relaxation of the integer programming formulation of ordinary dimension. The poset of a tree $T = (V,E)$ has ground set $V \cup E$ with ordering that puts edges above their endpoints. We derive tight upper bounds for the fractional dimension of posets of stars, binary trees, general trees, and infinite trees.

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For Corinne

Contents

List of Figures

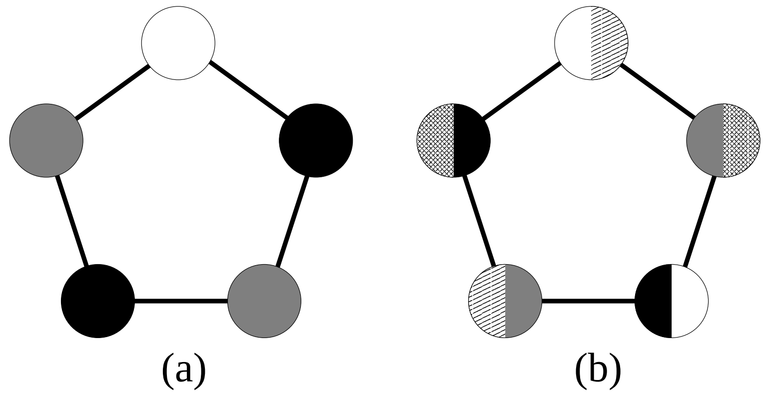

Figure 1: (a) illustrates a proper 3-coloring of $V(C_5)$. (b) illustrates a proper 5/2-fractional coloring of $V(C_5)$.

1 Introduction

Many graph invariants may be expressed as the solutions to constrained combinatorial optimization problems of the form "For the graph G, find the most (fewest) _____ such that _____." For example, chromatic number (denoted $\chi(G)$) is found by solving "Find the fewest colors necessary such that each vertex may be assigned a color and no two adjacent vertices are assigned the same color." For such invariants, we may define a corresponding *fractional* invariant, which is the solution to essentially the same problem, but we no longer require our solution to consist of whole pieces. In the case of fractional chromatic number (denoted $\chi_f(G)$), we solve the above problem, but now require only that each vertex get a *total* of one color, allowing a vertex to be colored 1/3 red, 1/3 blue and 1/3 green (for example). We still require that no amount of the same color be placed on adjacent vertices. If no vertex gets more than 1/3 of green, then green contributes 1/3 to the "total" number of colors used. To illustrate this, consider the 5-cycle, C_5. It is clear that $\chi(C_5) = 3$, as no odd cycle may be properly colored with only two colors (see Figure 1a). However, as illustrated in Figure

1b, we may color C_5 using 5 “half-colors”. No two adjacent vertices get any of the same color, and a total of 5/2 colors are used, so $\chi_f(C_5) \leq 2.5$. (As with chromatic number, presenting a proper fractional coloring only proves an upper bound on fractional chromatic number; in fact, 2.5 is the correct value for $\chi_f(C_5)$.) Because ordinary and fractional invariants may be more precisely expressed as integer and linear programs, respectively, the duals of these programs define dual ordinary and fractional invariants, which are used in calculating lower bounds. Other fractional invariants may be defined in a manner similar to the above, but as the majority of this work concerns itself with fractional chromatic number (and its dual, fractional clique number), we shall limit ourselves to defining just these two in this introduction. We limit our attention to simple graphs throughout, and further assume graphs to be finite unless we specify otherwise.

1.1 Definitions

To express chromatic number as an integer program, we first need to restate its definition. A coloring of the vertices of a graph G is said to be *proper* if no two adjacent vertices receive the same color[1]. Thus, the set of all vertices which receive a particular color is necessarily an independent set. So a proper coloring of $V(G)$ may also be thought of as a covering (or partition) of $V(G)$ by (into) independent sets, and $\chi(G)$ the fewest independent sets needed to cover $V(G)$. Based on this, we may formulate $\chi(G)$ as an integer program. The constraint matrix M will be the vertex/independent set incidence matrix, with rows indexed by $V(G) = \{v_1, \ldots, v_n\}$, and columns indexed by the independent subsets of

[1] Henceforth, “coloring” is to be understood as meaning “proper coloring”, since the subject of improper colorings does not come up in this dissertation.

$V(G)$: $\mathcal{I} = \{I_1, \ldots, I_m\}$. The i, j entry of M is a 1 exactly when $v_i \in I_j$, and is 0 otherwise. Then

$$\chi(G) \quad = \quad \min \mathbf{1}\cdot\mathbf{x} \quad \text{s.t. } M\mathbf{x} \geq \mathbf{1}, \;\; \mathbf{x} \geq 0, \;\; \mathbf{x} \in \mathbf{Z}^m \qquad \text{(IP)}$$

where **1** is the vector of all 1's of appropriate length. The dual integer program for this is

$$\omega(G) \quad = \quad \max \mathbf{1}\cdot\mathbf{y} \quad \text{s.t. } M^T\mathbf{y} \leq \mathbf{1}, \;\; \mathbf{y} \geq 0, \;\; \mathbf{y} \in \mathbf{Z}^n \qquad \text{(ID)}$$

This program finds the largest collection of vertices such that no two are in any independent set (i.e. all are adjacent), and therefore is a formulation of the clique number of G (denoted $\omega(G)$). We may now define the *fractional chromatic* and *fractional clique numbers* to be the solutions to the linear relaxations of the above integer programs:

$$\chi_f(G) \quad = \quad \min \mathbf{1}\cdot\mathbf{x} \quad \text{s.t. } M\mathbf{x} \geq \mathbf{1}, \;\; \mathbf{x} \geq 0, \;\; \mathbf{x} \in \mathbf{R}^m \qquad \text{(LP)}$$

$$\omega_f(G) \quad = \quad \max \mathbf{1}\cdot\mathbf{y} \text{ s.t. } M^T\mathbf{y} \leq \mathbf{1}, \;\; \mathbf{y} \geq 0, \;\; \mathbf{y} \in \mathbf{R}^n \qquad \text{(DP)}$$

Feasible solutions to these LPs are called *fractional colorings* and *fractional cliques*, respectively. Since these two programs are dual, the strong duality result of linear programming tells us that $\omega(G) \leq \omega_f(G) = \chi_f(G) \leq \chi(G)$ for any finite graph G.

There is a different, though equivalent, way of defining fractional chromatic number. A (proper) b-fold coloring of $V(G)$ is an assignment of a set of b colors to each vertex so that adjacent vertices receive disjoint color sets. We define the b-fold chromatic number, $\chi_b(G)$, to be the number of colors in the smallest proper b-fold coloring of $V(G)$. Because $\chi_b(G)$ is a non-negative sub-additive function[2] of b, we know that

$$\lim_{b\to\infty} \frac{\chi_b(G)}{b} = \inf_b \frac{\chi_b(G)}{b},$$

[2] By which we mean that $\chi_{b+c}(G) \leq \chi_b(G) + \chi_c(G)$.

and more importantly, we have the following.

Lemma 1.1 *For any finite graph G, $\chi_f(G) = \lim_{b\to\infty} \frac{\chi_b(G)}{b}$.*

Proof. [3] This result is most easily seen by thinking of $\chi_b(G)/b$ as a fractional coloring. Each color in a b-fold coloring still constitutes an independent set, so if we assign each such independent set a weight of $1/b$, every vertex gets covered by independent sets of total weight 1, and we have a fractional coloring of weight $\chi_b(G)/b$. Thus $\chi_f(G) \leq \chi_b(G)/b$ for all b, and $\chi_f(G) \leq \lim_{b\to\infty} \chi_b(G)/b$.

On the other hand, since $\chi_f(G)$ is the solution to a linear program with integer coefficients, there must be an optimal fractional coloring of $V(G)$ with rational weights. If we take b to be the least common denominator of all of these weights, then a reverse process (multiplying all weights by b) transforms this optimal fractional coloring into a proper b-fold coloring of weight $b \cdot \chi_f(G)$, so that $\chi_f(G) \geq \chi_b(G)/b$. Since the above limit is also an infimum, we are done. □

This also shows that $\chi_f(G) = \chi_b(G)/b$ for some b.

[3] Based on [11].

An equivalent way of comparing these two definitions is to write

$$\begin{aligned}\chi_b(G) &= \min \mathbf{1}\cdot\mathbf{x} \text{ s.t. } M\mathbf{x} \geq b\cdot\mathbf{1},\ \ \mathbf{x} \geq 0,\ \ \mathbf{x} \in \mathbf{Z}^m \\ \chi_b(G)/b &= \min \mathbf{1}\cdot\mathbf{x} \text{ s.t. } M\mathbf{x} \geq \mathbf{1},\ \ \mathbf{x} \in \{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}\}^m\end{aligned}$$

We can see intuitively that, as $b \to \infty$, the optimal solution to the latter program will approach that of the **(LP)** formulation of $\chi_f(G)$. Of course, the above proof only works in the case that G is finite. The proof for the infinite case is similar, however, and is presented in Chapter 2.

We may define the *b-fold clique number* of a graph in an analogous fashion. For a graph G, $\omega_b(G)$ is the size of the largest multiset of vertices with the property that no independent set of G contains more than b members (counting repetition) of this multiset. A more convenient formulation is one similar to the discrete programming formulation presented above for $\chi_b(G)$.

$$\begin{aligned}\omega_b(G) &= \max \mathbf{1}\cdot\mathbf{y} \text{ s.t. } M'\mathbf{y} \leq b\cdot\mathbf{1},\ \ \mathbf{y} \geq 0,\ \ \mathbf{y} \in \mathbf{Z}^n \\ \omega_b(G)/b &= \max \mathbf{1}\cdot\mathbf{y} \text{ s.t. } M'\mathbf{y} \leq \mathbf{1},\ \ \mathbf{y} \in \{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}\}^n\end{aligned}$$

This definition is presented for the sake of finding intermediate quantities between ω and ω_f. As such, our real interest lies in the quantity $\omega_b(G)/b$, which we henceforth refer to as *the b-clique number*[4] of G, or $\omega'_b(G)$. We refer to feasible solutions of the program defining $\omega'_b(G)$ as *b-cliques*. Similar to before, we have that $\lim_{b\to\infty} \omega'_b(G) = \omega_f(G)$; the proof is analogous to that for χ_f, and is found in Appendix A. Finally, we note that $\omega_f(G) \geq \omega'_b(G) \geq \omega(G)$ for all positive integers b.

[4] Not to be confused with the *b-fold* clique number.

Other definitions of fractional chromatic and clique numbers involve a game-theoretic approach and covering/packing problems on hypergraphs. A full treatment of these, and all the above material, may be found in [11].

1.2 Some Useful Facts

We start with a general observation.

Lemma 1.2 *The values of χ_f and ω_f don't change if we reformulate their definitions by taking $\mathcal{I}$ to be the set of all* maximal *independent sets.*

Proof. In an optimal solution of **(LP)**, if any non-maximal independent set gets positive weight, we may reassign that weight to a maximal independent set containing it. We have not decreased the weight on any vertex, and the value of our solution has remained the same.

In the dual program **(DP)**, if we only restrict the weight put on *maximal* independent sets by their vertices to be ≤ 1, then the weight put on any other independent set is necessarily also ≤ 1. □

In the following, we let $\alpha(G)$ denote the independence number of G, and let $[n] = \{1, 2, \ldots, n\}$.

Lemma 1.3 *For any finite graph G on n vertices, $\omega_f(G) \geq n/\alpha(G)$; if G is vertex transitive, then equality holds.*

Proof.[5] In the **(DP)** formulation of $\omega_f(G)$, let $y = \alpha(G)^{-1} \cdot \mathbf{1}$ (give every vertex weight $1/\alpha(G)$). No independent set can get total weight more than 1, so this solution is dual

[5] Based on [6].

feasible with value $n/\alpha(G)$, and we have our inequality.

Let G be vertex transitive and let $\mathbf{y}$ be an optimal solution to **(DP)**. Let aut(G) be the group of automorphisms on G, with $\pi \in \text{aut}(G)$. Let $\mathbf{y}_\pi = [y_{\pi(1)}, \ldots, y_{\pi(n)}]^T$. Note that $\mathbf{y}_\pi$ is also an optimal solution to **DP**, and, in general, any convex combination of optimal solutions is still an optimal solution. In particular,

$$\mathbf{y}^* = \frac{1}{|\text{aut}(G)|} \sum_{\pi \in \text{aut}(G)} \mathbf{y}_\pi$$

is an optimal solution. But since G is vertex transitive, $\mathbf{y}^*$ must have all entries equal, so $\mathbf{y}^* = a \cdot \mathbf{1}$ is an optimal solution for some a. But a can be no larger than $1/\alpha(G)$ lest a largest independent set get total weight greater than 1, so the value of this solution (i.e. $\omega_f(G)$) can be no larger than $n/\alpha(G)$, and we're done. $\square$

The construction of $\mathbf{y}^*$ above provides a more general insight into the work to follow. Even if we only have a transitive subset S of $V(G)$ (that is, for all $u, v \in S$ there exists $\pi \in \text{aut}(G)$ such that $\pi(u) = v$), we know that, without loss of generality, we may assign all vertices in S the same weight in a dual optimal solution. This observation is used repeatedly in the highly symmetric structures in this dissertation.

We next develop the class of circulant graphs, a generalization of cycles. The circulant graph $\langle S \rangle_n$ on n vertices with $S \subseteq \{1, 2, \ldots, \lfloor n/2 \rfloor\}$ is described as follows. Imagine the vertices of $\langle S \rangle_n$ to be equally spaced around a circle. Then if two vertices are d steps apart along the shorter arc of this circle, they are adjacent exactly if $d \in S$. Thus S is the set of "connection distances" in $\langle S \rangle_n$. Note that these graphs are vertex transitive, and that $\overline{\langle S \rangle_n} = \langle \bar{S} \rangle_n$, where $\bar{S} = \{1, 2, \ldots, \lfloor n/2 \rfloor\} - S$. In particular, we let $C_{n,m}$ denote $\langle \overline{[m-1]} \rangle_n$, so that two vertices are adjacent if they are at least m steps apart. The invariants

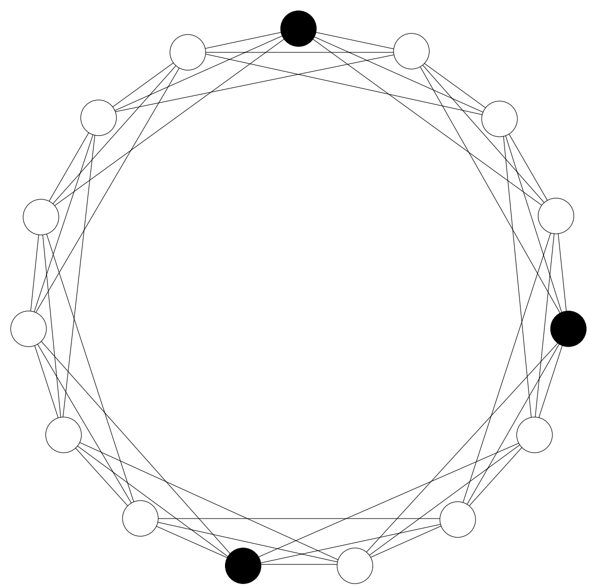

Figure 2: The graph $\overline{C_{15,4}} = \langle\{1,2,3\}\rangle_{15}$, with a maximum independent set of size $\lfloor\frac{15}{4}\rfloor = 3$ indicated.

$\alpha(C_{n,m})$ and $\omega(C_{n,m})$ are easily determined, giving us the following (see [6]):

Lemma 1.4 *$\alpha(C_{n,m}) = m$, $\omega(C_{n,m}) = \lfloor\frac{n}{m}\rfloor$, $\omega_f(C_{n,m}) = \frac{n}{m}$, and $\omega_f(\overline{C_{n,m}}) = \frac{n}{\lfloor n/m\rfloor}$.* □

We see that, for any rational $r \geq 2$, there is some circulant graph with $\omega_f = r$. We will let $C_{(r)}$ denote such a graph when we are only interested in the value of its fractional clique number.

Finally, we introduce graph sums and products. By the graph sum $G = H_1 \oplus H_2$ we mean that $V(G) = V(H_1) = V(H_2)$ and $E(G) = E(H_1) \cup E(H_2)$.[6] The graph product we wish to consider is the lexicographic product, denoted $G[H]$.[7] Here we have

[6]This notation often also includes the condition that $E(H_1) \cap E(H_2) = \emptyset$. We will sometimes make use of this extra condition, and will note specifically when we do so.

[7]Actually, we have two different uses for this notation. When G and H are distinct graphs, we mean lexicographic product. When $U \subseteq V(G)$, we take $G[U]$ to be the subgraph of G induced by U. Which of these two is intended will be clear in context.

$V(G[H]) = V(G) \times V(H)$, and $E(G[H])$ contains the edge $(u,x) \sim (v,y)$ iff either "$uv \in E(G)$" or "$u = v$ and $xy \in E(H)$". Roughly speaking, to create $G[H]$, we start with G and replace each vertex with a copy of H. We then draw all possible edges between two copies of H if the original vertices of G were adjacent.

Lemma 1.5 *For any graphs* G, H, $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$. $\square$

With one exception, the material of this introduction may be found in [11] and [6]. Proofs are reproduced here to serve as templates for the work to follow. The proof of Lemma 1.5 for finite graphs may be found in [11]; the proof for infinite graphs may be found in Appendix A of this dissertation.

1.3 Overview of Results

In Chapter 2, we consider the fractional analog of the Erdös-de Bruijn Theorem, which states that the chromatic number of an infinite graph is the supremum of the chromatic numbers of all its finite subgraphs. Leader [8] shows that this is not the case for fractional chromatic number. Let $\overline{\chi_f}(G)$ be the supremum of χ_f of all of G's finite subgraphs. Leader constructs infinite graphs with $\chi_f(G) = \infty$ and $\overline{\chi_f}(G)$ any rational value > 2. We answer an open question posed therein by constructing a class of infinite graphs with $\overline{\chi_f}(G) < \chi_f(G) < \infty$. We also show that $\omega_f(G) = \overline{\chi_f}(G)$, which then proves that ω_f and χ_f are not, in general, equal for infinite graphs.

In Chapter 3, we define and analyze fractional Ramsey numbers. These are defined by simply replacing ω with ω_f in the ordinary Ramsey number definition. Whereas ordinary Ramsey numbers are notoriously difficult to calculate and grow exponentially, we derive an exact formula for fractional Ramsey numbers and therein show that they grow quadratically. We also fractionalize the multi-color version of Ramsey numbers, and derive some bounds and specific results there. Finally, we discuss several generalizations of fractional Ramsey number.

In Chapter 4, we consider the fractional dimension of partially ordered sets, where an incomparable pair/realizer definition of fractional dimension is analogous to the vertex/independent set definition of fractional chromatic number. We study posets derived from trees, where the ground set consists of vertices and edges, and an edge is greater than its endpoints. We calculate tight upper bounds for the fractional dimension of posets of stars, binary trees and general trees. We also show that the fractional dimension of any poset of an infinite tree with unbounded degree is 3.

2 The Fractional Chromatic Gap

As previously noted, $\chi_f(G) = \omega_f(G)$ for any finite graph G. This result follows from the strong duality of linear programs. Since there is no such duality result for infinite linear programs, it is reasonable to ask, “Does this equality still hold?” In general it does not, and the proof of this comes in conjunction with an open problem presented by Leader [8], which shall be addressed after presenting a few definitions.

To start, we need to alter the **(LP)** formulation of $\chi_f(G)$ to remove the linear programming language and accommodate infinite graphs. In this we follow Leader’s notation. We still let $\mathcal{I}$ represent the set of independent sets of G, and then define a *fractional coloring* of G to be a mapping $f : \mathcal{I} \to [0, 1]$ such that for each $v \in G$ we have $\sum_{I \in \mathcal{I}: v \in I} f(I) \geq 1$. The weight of this coloring is $w(f) = \sum_{I \in \mathcal{I}} f(I)$. Leader’s definition of fractional chromatic number (which we call $\chi^*(G)$ for now) is

$$\chi^*(G) = \inf\{w(f) : f \text{ a fractional coloring of } G\}.$$

Note that, if G is finite, this is equivalent to the **(LP)** formulation. Further, $\chi^*(G)$ is well-defined in the case that G is infinite. For the time being, we reserve the χ_f notation to represent the b-fold formulation, which is still clearly valid for infinite graphs. What is not clear is that these two formulations are still equivalent in the infinite case. We shall show that they are.

We similarly modify the definition of fractional clique number: a *fractional clique* of G is a mapping $g : V(G) \to [0, 1]$ such that for each $I \in \mathcal{I}$ we have $\sum_{v \in I} g(v) \leq 1$. The weight of this mapping is $w(g) = \sum_{v \in V(G)} g(v)$, and fractional clique number is

$$\omega^*(G) = \sup\{w(g) : g \text{ a fractional clique of } G\}.$$

Again, this is equivalent to **(DP)** if G is finite[8].

Finally, we define

$$\overline{\chi_f}(G) = \sup\{\chi_f(H) : H \text{ a finite subgraph of } G\},$$

The Erdös-de Bruijn theorem [4] states that the ordinary chromatic number of an infinite graph equals the supremum of the chromatic numbers of all its finite subgraphs. Zhu [17] asked if this was the case for fractional chromatic number, and Leader [8] answered in the negative by constructing graphs G with $\chi_f(G) = \infty$ and $\overline{\chi_f}(G)$ any rational number larger than 2. He then asked if there exists an infinite G for which $\overline{\chi_f}(G) < \chi_f(G) < \infty$.

In this chapter, we construct a class of such graphs. Further, by proving that $\omega_f(G) = \overline{\chi_f}(G)$, we show that the strong duality result for fractional chromatic number is, in general, false for infinite graphs.

2.1 Two Equalities for Infinite Graphs

The proof that $\chi^*(G) = \lim_{b\to\infty} \chi_b(G)/b$ for finite G is fairly simple, but depends both on there being only a finite number of independent sets, and on the fact that linear programs with integer coefficients have rational solutions. For infinite graphs, we have neither of these. However, we may find rational and finite fractional colorings arbitrarily close to any fractional coloring, and this is sufficient.

[8]Since we do not use the b-fold formulation of ω_f here, we henceforth use this notation in place of ω^*. A proof that these two are equal appears in Appendix A.

Theorem 2.1 *For any infinite graph G, $\chi^*(G) = \chi_f(G)$.*

Proof. The proof from Lemma 1.1 that $\chi^*(G) \leq \lim_{b\to\infty} \chi_b(G)/b$ still works in the infinite case[9]: any b-fold coloring of value $\chi_b(G)$ is transformed into a fractional coloring of value $\chi_b(G)/b$, thereby proving that $\chi^*(G) \leq \chi_f(G)$.

Since $\chi^*(G) \leq \chi_f(G) \leq \chi(G)$, and Leader[8] showed that $\chi(G) = \infty$ implies $\chi^*(G) = \infty$, then we are done if $\chi^*(G) = \infty$. So we restrict our attention to the case of $\chi^*(G) < \infty$.

To proceed with the other inequality, for any given $\epsilon > 0$, we wish to find a positive integer b_0 such that $\chi_b(G)/b \leq \chi^*(G) + \epsilon$ for all $b \geq b_0$. If we can accomplish this, then $\lim_{b\to\infty} \chi_b(G)/b \leq \chi^*(G)$, and we are done. We will start with a fractional coloring f of weight sufficiently close to $\chi_f(G)$, then make two approximations of it: (i) restrict f to being positive on only a finite number of independent sets, and (ii) find sufficiently large b so that f may be rounded up to multiples of $1/b$ with negligible addition of total weight. We may then use the method of Lemma 1.1 to convert f into a proper b-fold coloring.

(i) Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{2}(1 + \chi^*(G) + \frac{\epsilon}{2})^{-1}$, so that $\frac{\chi^*(G)+\delta}{1-\delta} = \chi^*(G) + \epsilon/2$. Now, choose a fractional coloring f with $\chi^*(G) < w(f) \leq \chi^*(G) + \delta$. Since $w(f)$ is a (possibly) infinite sum of finite value, we may find finite partial sums arbitrarily close to this value. More specifically, there exists a finite $\mathcal{I}' \subset \mathcal{I}$ (where $\mathcal{I}$ are the independent sets of G) such that

$$\chi^*(G) \leq \sum_{I\in\mathcal{I}'} f(I) \leq \sum_{I\in\mathcal{I}} f(I) = w(f) \leq \chi^*(G) + \delta \ .$$

[9] Note, however, that what we called "χ_f" in that proof we are now calling "χ^*".

Let $n = |\mathcal{I}'|$, and let f' be the weighting of $\mathcal{I}$ by f restricted to $\mathcal{I}'$, so that $w(f') = \sum_{I \in \mathcal{I}'} f(I)$. Since $w(f')$ is within δ of $w(f)$, f' must give each vertex of G weight at least $1-\delta$. We now define f'' by multiplying each $f'(I)$ by $1/(1-\delta)$, so that f'' gives each vertex weight at least 1, and is thus a valid fractional coloring. Further, $w(f'') = w(f')/(1-\delta) \leq \frac{\chi^*(G)+\delta}{1-\delta} = \chi^*(G) + \epsilon/2$ by our choice of δ.

(ii) Now, choose $b > 2n/\epsilon$, and create the fractional coloring f_b by rounding $f''(I)$ up to the nearest multiple of $1/b$ for each $I \in \mathcal{I}'$. We have only added weight, so f_b is still a valid fractional coloring, and we have added at most $|\mathcal{I}'|/b = n/b < \epsilon/2$ weight to $w(f'')$. So $w(f_b) \leq w(f'') + \epsilon/2 \leq \chi^*(G) + \epsilon$. We now have a rational fractional coloring with common denominator b using only a finite collection of independent sets. For each $I \in \mathcal{I}'$, if we associate $b \cdot f_b(I)$ distinct colors (and apply them to each $v \in I$), then every vertex in G gets (at least) b colors. Since each color constitutes an independent set, we have created a proper b-fold coloring. We have used $b \cdot w(f_b)$ colors, and so $\chi_b(G)/b \leq (b \cdot w(f_b))/b \leq \chi^*(G) + \epsilon$. Further, this works for *any* $b > 2n/\epsilon$, so we have our desired result. □

We next address the relation between ω_f and χ_f for infinite graphs with the following.

Theorem 2.2 *If G is an infinite graph, then $\omega_f(G) = \overline{\chi_f}(G)$.*

Proof. Clearly $\omega_f(G) \geq \omega_f(H) = \chi_f(H)$ for any finite subgraph H of G, so $\omega_f(G) \geq \overline{\chi_f}(G)$.

On the other hand, if $\omega_f(G) > \overline{\chi_f}(G)$, then G has a fractional clique g with $w(g) > \overline{\chi_f}(G)$. Since $w(g)$ is a (possibly) infinite sum, we may find a finite partial sum arbitrarily close to $w(g)$; specifically, we may find one greater than $\overline{\chi_f}(G)$. But this finite partial sum is simply a weighting of a finite subset U of $V(G)$. Let g' be g restricted to U. Any independent set of $G[U]$ (the finite subgraph induced by U) must also be an independent set of G, and so g' is a fractional clique of $G[U]$. But then $\overline{\chi_f}(G) < w(g') \leq \omega_f(G[U]) = \chi_f(G[U])$, which is a contradiction. Thus $\omega_f(G) > \overline{\chi_f}(G)$ must be false, and our result follows. □

Since we have examples of infinite graphs for which $\overline{\chi_f}(G) < \chi_f(G)$ (see Leader [8] and the following section), we know that, unlike the case of finite graphs, ω_f and χ_f can differ in infinite graphs.

2.2 Construction of Graphs with $\overline{\chi_f}(G) < \chi_f(G) < \infty$

We first define graphs G^n and $G^{n,m}$. We let $V(G^n) = L \cup R$, where L is the set of positive integers $\mathbf{N} = \{1, 2, 3, \ldots\}$. For every size n subset N of L we put a distinct copy of K_n in R, and adjoin each vertex of this K_n to a distinct vertex of N. Thus every vertex in R is adjacent to exactly one vertex in L, and to $n-1$ vertices in R. $G^{n,m}$ is defined identically, except that $L = [m] = \{1, 2, \ldots, m\}$. $G^{n,m}$ may alternately described by starting with a complete n-regular hypergraph on m vertices, and then forming a graph by replacing each hyperedge with a new, distinct copy of K_n, and adjoining each vertex of this new K_n to a distinct vertex from the original hyperedge.

We now define the graphs $G^n_{r,s}$ and $G^{n,m}_{r,s}$ by replacing the vertices of G^n and $G^{n,m}$, respectively, with circulant graphs[10]; we shall refer to these circulant graphs within $G^n_{r,s}$ and $G^{n,m}_{r,s}$ as "nodes." To be precise, we replace each vertex in L with a $C_{(r)}$, each vertex in R with a $C_{(s)}$, and all possible edges are drawn between two nodes exactly when their original vertices were adjacent in G^n or $G^{n,m}$. Alternately, recalling the lexicographic product, L may be thought of as $\overline{K_\infty}[C_{(r)}]$ or $\overline{K_m}[C_{(r)}]$ (in $G^n_{r,s}$ or $G^{n,m}_{r,s}$, respectively), and

[10] Recall that $C_{(r)}$ is a circulant graph with $\chi_f(C_{(r)}) = r$.

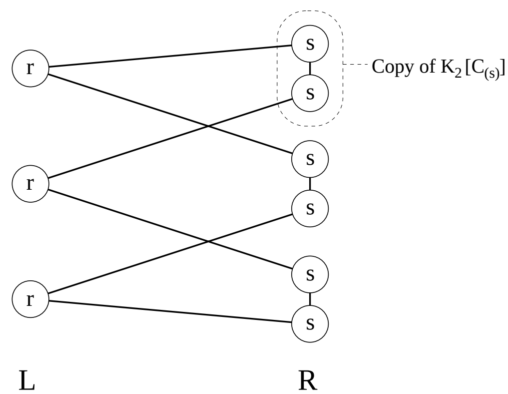

Figure 3: The graph $G^{2,3}_{r,s}$. $n = 2$ since R is comprised of disjoint $K_2[C_{(s)}]$'s. $m = 3$ is the number of nodes in L. The nodes shown in L and R represent copies of $C_{(r)}$ and $C_{(s)}$, respectively. The edges shown actually represent all possible edges between nodes.

R as a collection of disjoint copies of $K_n[C_{(s)}]$. Figure 3 shows $G^{2,3}_{r,s}$.

If $U \subset V(G)$, we let $G[U]$ denote the subgraph of G induced by U. From Lemma 1.5 and our observations about L and R, we see that

$$\begin{array}{rcccccl} \chi_f(G^n_{r,s}[L]) & = & \chi_f(G^{n,m}_{r,s}[L]) & = & \chi_f(C_{(r)}) & = & r \\ \chi_f(G^n_{r,s}[R]) & = & \chi_f(G^{n,m}_{r,s}[R]) & = & \chi_f(K_n[C_{(s)}]) & = & ns\ , \end{array}$$

and similarly,

$$\begin{array}{rcccccl} \chi_b(G^n_{r,s}[L]) & = & \chi_b(G^{n,m}_{r,s}[L]) & = & \chi_b(C_{(r)}) & \geq & br \\ \chi_b(G^n_{r,s}[R]) & = & \chi_b(G^{n,m}_{r,s}[R]) & = & \chi_b(K_n[C_{(s)}]) & \geq & bns\ . \end{array}$$

A little thought reveals that every finite subgraph of $G^n_{r,s}$ must also be a subgraph of some $G^{n,m}_{r,s}$, so $\overline{\chi_f}(G^n_{r,s}) = \lim_{m\to\infty} \chi_f(G^{n,m}_{r,s})$ (since $H \subset G$ implies $\chi_f(H) \leq \chi_f(G)$). We are now ready to start our calculations.

Theorem 2.3 $\chi_f(G^n_{r,s}) = r + ns$.

Proof. Since $\chi_f(G^n_{r,s}[L]) = r$ and $\chi_f(G^n_{r,s}[R]) = ns$, $\chi_f(G^n_{r,s}) \leq r + ns$ is immediate.

To prove equality, it suffices to show that $\chi_b(G^n_{r,s}) \geq (r + ns)b$ for all $b \in \mathbf{N}$. This, in turn, is true if, for all $b \in \mathbf{N}$, there exists $m \in \mathbf{N}$ such that $\chi_b(G^{n,m}_{r,s}) \geq (r + ns)b$ (since $\chi_b(G^n_{r,s}) \geq \chi_b(G^{n,m}_{r,s})$ for all b and m).

Fix $b \in \mathbf{N}$. Whatever $\chi_b(G^n_{r,s})$ is (say, c), any optimal b-fold coloring of any $G^{n,m}_{r,s}$ will use no more than c colors. Suppose that $C_{(r)}$ has a vertices, and consider the set of colors put on any copy of $C_{(r)}$ in L. We are assigning b colors to each vertex in that node, but more to the point, we are assigning no more than ab of a finite set of c colors to this node. There are only a finite number of ways to do this, so if we color enough nodes, we will necessarily have nodes with identical color sets. What's more, with a, b and c fixed, if we take m (the number of nodes in L) large enough, we can guarantee that at least n nodes in L will share identical color sets in *any* optimal b-fold coloring of $V(G^{n,m}_{r,s})$ (this observation follows from a simple pigeon-hole argument). In such a b-fold coloring of such a $G^{n,m}_{r,s}$, consider the $K_n[C_{(s)}]$ in R which corresponds to these n identically colored nodes in L. Since every vertex of every node of this $K_n[C_{(s)}]$ is adjacent to these same colors, a completely disjoint color set must be used to color these vertices of R. So at best, we can color the nodes of L using $\chi_b(C_{(r)}) \geq rb$ colors, and some copy of $K_n[C_{(s)}]$ with $\chi_b(K_n[C_{(s)}]) \geq nsb$ *different* colors. Thus $\chi_b(G^{n,m}_{r,s}) \geq (r + ns)b$, as desired. □

We next turn our attention to the problem of finding $\overline{\chi_f}(G^n_{r,s})$, which in turn requires computing $\chi_f(G^{n,m}_{r,s})$. This can't be done exactly in most cases, but can be expressed in terms of a root of a simple polynomial. For convenience, we define Q_f to be the set of all rationals in $\{1\} \cup [2, \infty)$, that is, the set of all fractional chromatic numbers of finite graphs (see [11]).

Theorem 2.4 *For any $n \in \mathbf{N}$ and $r, s \in Q_f$, let p_0 be the real root of $rx^n + nsx - r = 0$ in $(0, 1)$. Then $\overline{\chi_f}(G^n_{r,s}) = r/p_0$.*

Proof. Fix $n \in \mathbf{N}$ and $r, s \in Q_f$. We first observe that $f(x) = rx^n + nsx - r$ has a single real root in $(0, 1)$, since $f'(x) = rnx^{n-1} + ns > 0$ for $x \geq 0$, $f(0) = -r$ and $f(1) = ns$.

Since $\overline{\chi_f}(G^n_{r,s}) = \lim_{m\to\infty} \chi_f(G^{n,m}_{r,s})$, it suffices to prove the following two inequalities:

(I) $\chi_f(G^{n,m}_{r,s}) \leq r/p_0$ for all $m \in \mathbf{N}$

(II) $\lim_{m\to\infty} \omega_f(G^{n,m}_{r,s}) \geq r/p_0$

since $\omega_f(G) = \chi_f(G)$ for any finite G.

(I) $\chi_f(G^{n,m}_{r,s}) \leq r/p_0$

Fix $m \in \mathbf{N}$, and consider the maximal independent sets of $G^{n,m}_{r,s}$. We may fully describe any such set with a single parameter p (up to a few irrelevant decisions[11]). Denote such an independent set by I_p, where p is the fraction of nodes in L which have at least one vertex in I_p. We say that such a node is *covered* by I_p. Putting even a single vertex from such an L node in I_p excludes from I_p anything in R adjacent to that node. So if I_p is to be maximal, from every covered node in L we must put in I_p a maximal independent set from that node (which is a copy of $C_{(r)}$). Since all maximal independent sets of $C_{(r)}$ are the same up to isomorphism, our choice of this set is irrelevant. Next consider any of one of the disjoint copies of $K_n[C_{(s)}]$ in R. Each node in this subgraph has a "matched" node in L, and we may only include in I_p vertices from a node of R if its matching node in L is not covered by I_p (every vertex in a node of R is adjacent to every vertex in its matching node in L). So in choosing vertices from this $K_n[C_{(s)}]$ for I_p, we need only consider nodes whose matching nodes in L are not covered. Further, once we include in I_p a vertex from any one node of this $K_n[C_{(s)}]$, we exclude all its other nodes from I_p, since this vertex is adjacent to every vertex in every other node of this $K_n[C_{(s)}]$. So I_p intersects at most one node of any $K_n[C_{(s)}]$. The choice of which "match-uncovered" node is irrelevant for our purposes. Once we have selected the node, since we want I_p to be maximal, we must take a maximal independent set from that copy of $C_{(s)}$. Again, which one is irrelevant up to isomorphism. Thus, by specifying only p, we have (excepting a few equivalent choices) fully described what the maximal independent sets I_p must be.

[11] We would like to say "up to isomorphism", but this is not strictly true. However, it behaves this way for our purposes.

Since these are our only maximal independent sets, they will be the only independent sets to receive positive weight in our optimal fractional coloring. In particular, we wish to limit ourselves to weighting sets with the "best" value of p. This value will be the one for which all such I_p cumulatively place the same total weight on vertices in L and R. To this end, let p be fixed, and imagine I_p to be a random variable; specifically, an independent set chosen uniformly and at random from the finite number of maximal independent sets with parameter p. We then wish to equate $\Pr\{v \in I_p \mid v \in L\}$ and $\Pr\{v \in I_p \mid v \in R\}$. Note that if we choose a maximal independent set uniformly and at random in $C_{(r)} = C_{a:b}$, the probability that a given vertex is in this set is exactly $b/a = 1/r$. Since p is the fraction of nodes in L which are covered by I_p,

$$\begin{aligned}\Pr\{v \in I_p \mid v \in L\} &= \Pr\{v \text{ is in a node covered by } I_p\} \cdot \\ &\qquad \Pr\{v \text{ is in an independent set of its node}\} \\ &= p/r.\end{aligned}$$

Given $v \in R$, for the event $v \in I_p$ to occur, three conditions must be met: (i) the matching node in L of v's node must not be in I_p, (ii) v's node must be chosen from among all such "unmatched" nodes in its copy of $K_n[C_{(s)}]$, and (iii) v must be in an independent set chosen from its node. Now, $\Pr\{(i)\} = 1 - p$, and $\Pr\{(iii)\} = 1/s$, but $\Pr\{(ii)\}$ is conditional on the number of other unmatched nodes in this copy of $K_n[C_{(s)}]$. We will let the index k count the total number of such unmatched nodes (including v's), and K will be $k-1$ (the number of *other* unmatched nodes). $\Pr\{K = k-1\}$ for any fixed value of k is described as follows: if we have a huge (size m) pool of objects, from which a fraction p are being chosen ("matched"), and we consider a specific collection of

$n-1$ of these objects (before choosing), what is the probability that exactly $n-k$ of these will be chosen (so that $k-1$ are "unmatched")? The answer to this is not easy; because we are choosing from a finite sample, if one of our specified objects is chosen, it affects the probability that others are chosen. However, when $m >> n$, this affect is negligible, and we can approximate this process by letting each of our $n-1$ specified objects be chosen independently with probability p. That is, as m gets large we may approximate this probability by a Bin$(n-1,p)$ distribution, and we get

$$\begin{aligned}\Pr\{(ii)\} &= \sum_{k=1}^{n} \Pr\{(ii) \mid K = k-1\} \cdot \Pr\{K = k-1\} \\ &\approx \sum_{k=1}^{n} \left(\frac{1}{k}\right) \cdot \left(\binom{n-1}{k-1} (1-p)^{k-1} p^{n-k} \right)\end{aligned}$$

and

$$\begin{aligned}\Pr\{v \in I_p \mid v \in R\} &= \Pr\{(i)\} \cdot \Pr\{(ii)\} \cdot \Pr\{(iii)\} \\ &\approx \frac{1}{ns} \sum_{k=1}^{n} \binom{n}{k} (1-p)^k p^{n-k} \\ &\approx \frac{1}{ns} (1-p^n).\end{aligned}$$

Now we set $\Pr\{v \in I_p \mid v \in L\} = \Pr\{v \in I_p \mid v \in R\}$, and get $rp^n + nsp - r = 0$. If p_0 is the real root of this polynomial in $(0,1)$, then both of the above probabilities are equal to p_0/r. So each $v \in G_{r,s}^{n,m}$ occurs in exactly a fraction p_0/r of the maximal independent sets with parameter p_0. If we distribute total weight r/p_0 equally among all such independent sets, then each vertex in $G_{r,s}^{n,m}$ will be in independent sets of total weight exactly $(r/p_0)(p_0/r) = 1$. Thus we have created a valid fractional coloring with total weight r/p_0.

Of course, p_0 is liable to be irrational, and in any case not a multiple of $1/m$, so we can't actually choose exactly a fraction p_0 of the nodes in L to be covered by an I_p. However, as $m \to \infty$, we may choose p arbitrarily close to p_0. Also, recall that the value of $\Pr\{K = k-1\}$ was only approximated by a binomial distribution. But again, this becomes arbitrarily close to correct as $m \to \infty$, and so r/p_0 *will* be an upper bound on $\lim_{m\to\infty} \chi_f(G^{n,m}_{r,s})$. Since $G^{n,m}_{r,s} \subset G^{n,m+1}_{r,s}$, we know $\chi_f(G^{n,m}_{r,s})$ must be a non-decreasing function of m. Thus r/p_0 must actually be an upper bound on $\sup_{m\in\mathbf{N}} \chi_f(G^{n,m}_{r,s})$, and we're done.

(II) $\lim_{m\to\infty} \omega_f(G^{n,m}_{r,s}) \geq r/p_0$

Given $G^{n,m}_{r,s}$ as before, we wish to produce a fractional clique g with weight r/p_0 (more correctly, a sequence of fractional cliques with weights which will approach r/p_0 as $m \to \infty$). We assign total weight α to L, divided evenly among these vertices (each vertex gets weight $\alpha/|L|$), and weight β to R, also evenly distributed, so that $w(g) = \alpha + \beta$. We require that g put weight at most 1 in any independent set. Since the I_p's are the only maximal independent sets in G, we need only worry about the weight on them. The total weight put on any I_p is

$$w_{\alpha,\beta}(p) = \alpha \cdot \frac{|I_p \cap L|}{|L|} + \beta \cdot \frac{|I_p \cap R|}{|R|} \ .$$

But $\frac{|I_p \cap L|}{|L|}$ is just the previously calculated $\Pr\{v \in I_p \mid v \in L\}$, and similarly for R, so

$$w_{\alpha,\beta}(p) = \alpha \frac{p}{r} + \beta \frac{1-p^n}{ns} \ .$$

Now, let us set

$$\alpha = \frac{r}{p_0}\left(\frac{rp_0^{n-1}}{s + rp_0^{n-1}}\right), \quad \beta = \frac{r}{p_0}\left(\frac{s}{s + rp_0^{n-1}}\right),$$

where p_0 is the real root of $rp^n + nsp - r = 0$ in $(0,1)$. This selection of α and β gives us the following:

(i) $\alpha + \beta = r/p_0$.

(ii) Taking derivatives of $w_{\alpha,\beta}(p)$ with respect to p gives

$$w'_{\alpha,\beta}(p) = \frac{\alpha}{r} - \frac{\beta}{s}p^{n-1}\ , \quad w'_{\alpha,\beta}(p_0) = 0\ ; \quad w''_{\alpha,\beta}(p) = -\frac{\beta(n-1)}{s}p^{n-2} \leq 0 \text{ for } p \geq 0.$$

From the above, we see that $w_{\alpha,\beta}(p)$ attains its maximum value on $p \in [0,1]$ at p_0.

(iii) Since $(1 - p_0^n)/ns = p_0/r$, we have $w_{\alpha,\beta}(p_0) = (p_0/r)(\alpha + \beta) = 1$.

This shows that 1 is the largest value given by g to any I_p, and thus to any independent set of G. Thus g is a valid fractional clique, and has total weight r/p_0.

Again, we must be careful. As noted at the end of part(I), Pr$\{v \in I_p \mid v \in R\}$ is not quite $(1 - p^n)/ns$. However, it will approach this value as $m \to \infty$. So while no $G_{r,s}^{n,m}$ will actually have fractional clique number equal to r/p_0, as m gets large we may construct fractional cliques of $G_{r,s}^{n,m}$ with values arbitrarily close to r/p_0. Since fractional clique number is a maximization LP, these values provide a lower bound on ω_f, so $\lim_{m\to\infty} \omega_f(G_{r,s}^{n,m}) \geq r/p_0$ as desired. □

2.3 The Behavior of $\chi_f(G)$ vs. $\overline{\chi_f}(G)$

We have established the existence of graphs for which $\overline{\chi_f}(G) < \chi_f(G) < \infty$. Next we might well ask, "For *which* $x < y < \infty$ does there exist a graph with $x = \overline{\chi_f}(G)$ and $y = \chi_f(G)$?" We are only concerned with such (x,y) pairs where $2 < x < y$, since $\chi_f(G) = 2$ implies G is bipartite, even for infinite graphs.

We define

$$
\begin{aligned}
S_n &= \{(x,y)\in \mathbf{R}^2 : x=\overline{\chi_f}(G^n_{r,s}),\ \ y=\chi_f(G^n_{r,s}) \text{ for some } r,s\in \mathbf{Q}_f\} \\
&= \{(x,y)\in \mathbf{R}^2 : \exists r,s\in \mathbf{Q}_f \text{ s.t. } \frac{r}{x}\in(0,1), \\
&\qquad \left(\frac{r}{x}\right)^n+\frac{ns}{x}-1=0,\ \ y=r+ns\} \ .
\end{aligned}
$$

Note that each point in S_n is generated by an ordered pair $(r,s)\in \mathbf{Q}_f^2$. Because the definition of S_n generally involves a high order polynomial, it is difficult to describe this set more precisely. S_2, however, only involves a quadratic, and we may solve for x in terms of r and s:

$$
x=\frac{r^2}{-s+\sqrt{r^2+s^2}} \ .
$$

Setting $(r,s)=(1,1)$ generates $(x,y)=(1+\sqrt{2},3)$, which is an isolated point of S_2. Holding one of r or s fixed at 1 and letting the other one increase from 2 generates a curve which quickly approaches the line $y=x+1$. Finally, $\{(r,s):r,s\geq 2\}$ generates a solid[12], roughly cone-shaped region with point at $(x,y)=(2+2\sqrt{2},6)$ (see Figure 4). For higher values of n, we may use root-finding software to plot S_n, and we see a set of the same general shape as S_2. Also, we may bound the ratio x/y for S_2 and for all S_n.

Theorem 2.5 $\frac{\chi_f(G^2_{r,s})}{\overline{\chi_f}(G^2_{r,s})}\leq \frac{5}{4}$. *This bound is tight.*

Proof. We have just seen that

$$
\overline{\chi_f}(G^2_{r,s}) \ = \ x \ = \ \frac{r^2}{-s+\sqrt{r^2+s^2}}.
$$

[12]The region as described is not actually solid, as it is generated only by rational r and s. However, if our goal is to cover as much of the plane as possible, we may construct G to be a disjoint sequence of $G^{\,n}_{r,s}$ in which the r's and s's approach any desired real limits. It is easy to show that $\overline{\chi_f}(G)$ and $\chi_f(G)$ actually take on their expected values as indicated by these limits.

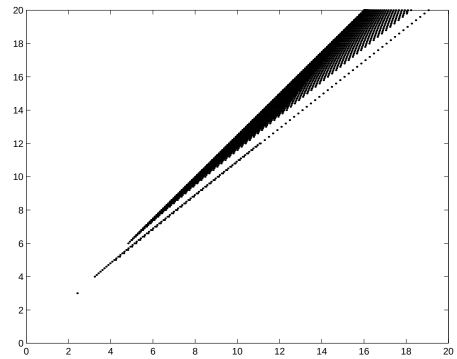

Figure 4: The region S_2.

If we solve for s, we get $s = \frac{1}{2}\left(x - \frac{r^2}{x}\right)$. We now fix x, and see that

$$\begin{aligned} \chi_f(G^2_{r,s}) &= y = r + 2s = r + x - \frac{r^2}{x} \\ \frac{dy}{dr} &= 1 - \frac{2r}{x} = 0 \quad \text{at} \quad r = x/2 \\ \frac{d^2y}{dr^2} &= -2/x < 0 \end{aligned}$$

If we take $x = \overline{\chi_f}$ to be a fixed value, we may still vary the value of $y = \chi_f$ by varying r and s. And the above shows that, as a function of r, y is maximized at $r = x/2$. At this value, $y = \frac{5}{4}x$, that is, $\chi_f(G^2_{r,s}) = \frac{5}{4}\overline{\chi_f}(G^2_{r,s})$. This is the largest χ_f can be relative to $\overline{\chi_f}$, and our construction guarantees that this ratio is actually achieved. □

Theorem 2.6 $\frac{\chi_f(G^n_{r,s})}{\overline{\chi_f}(G^n_{r,s})} \leq 2$ *for any integer* $n \geq 2$ *and* $r, s \in Q_f$. *Further, if we keep* $r = ns$, *then this bound is tight as* $n \to \infty$.

Proof. Since any $G^{n,m}_{r,s}$ contains $C_{(r)}$ and $K_n[C_{(s)}]$ as subgraphs, we must have $\chi_f(G^{n,m}_{r,s}) \geq \max\{r, ns\}$, so

$$\chi_f(G^n_{r,s}) = r + ns \leq 2\max\{r, ns\} \leq 2\chi_f(G^{n,m}_{r,s}),$$

which proves the first half of our claim.

Next, set $r = ns$, so that p_0 is the root of $x^n + x - 1 = 0$ in $(0,1)$. Let $n \to \infty$. Then $p_0 \to 1$ since $x^n \to 0$ for any x in $(0,1)$. So we have $\overline{\chi_f}(G^n_{r,s}) = r/p_0 \to r$ and $\chi_f(G^n_{r,s}) = r + ns = 2r$. □

We may now define $S = \cup_{n=2}^{\infty} S_n$ to be the region of the plane covered by ordered pairs of the form $(\overline{\chi_f}(G^n_{r,s}), \chi_f(G^n_{r,s}))$. Again, if our goal is to cover as much of the plane as possible, we may resort to one more trick: recall from Lemma 1.5 that $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$, and note that$\overline{\chi_f}(G[H]) = \overline{\chi_f}(G)\overline{\chi_f}(H)$ follows immediately from this. Since any two points in S represent two known graphs, we may take their lexicographic product to get a graph with $(\overline{\chi_f}(G), \chi_f(G))$ equal to the component-wise product of the two points in S. More generally, by taking multiple graph products, we may now cover the following region of the plane:

$$S' = \{(\prod_{i=1}^{k} x_i, \prod_{i=1}^{k} y_i) \in \mathbf{R}^2 : (x_i, y_i) \in S \text{ for } i = 1, \ldots, k,\ k \in \mathbf{N}\} \quad .$$

In particular, for $(x, y) \in S'$, the ratio y/x is no longer bounded, since $(x^k, y^k) \in S'$ for any integer k and $(x, y) \in S$. However, attaining large ratios also requires large values of x and y. So, for instance, while $(n, r, s) = (2, 4, 3)$ gives the point $(8, 10) \in S_2$, and Leader [8] constructs a graph with $(\overline{\chi_f}(G), \chi_f(G)) = (8, \infty)$, it is unknown whether or not a graph exists with $(\overline{\chi_f}(G), \chi_f(G)) = (8, 80)$ (for instance). So while much of the plane has been covered herein, we still have the open problem "Given any real x and y with $y > x > 2$, does there exist G with $\overline{\chi_f}(G) = x$ and $\chi_f(G) = y$?"

3 Fractional Ramsey Numbers

Since the definition of Ramsey numbers makes use of the clique number of graphs, we may define fractional Ramsey numbers simply by substituting fractional clique number into this definition. Recall that the Ramsey arrow notation "$n \to (k,l)$" means that "For any red/blue-coloring of the edges of K_n, there must a either a red k-clique or a blue l-clique." More precisely, whenever $K_n = H_1 \oplus H_2$, then we must have $\omega(H_1) \geq k$ or $\omega(H_2) \geq l$. This edge-decomposition of K_n may be thought of as an edge 2-coloring. The *Ramsey number* $r(k,l)$ is the least positive integer n for which this statement is true. Very few exact values for $r(k,l)$ are known. For instance, $r(5,5)$ is only known to be somewhere between 43 and 55. And while the growth rate of $r(k,k)$ is known to be an exponential function of k, the best known lower and upper bounds on this growth are (roughly) $\sqrt{2}^k$ and 4^k, respectively.

Now, we may define the fractional Ramsey arrow notation $n \xrightarrow{f} (x,y)$ to mean that, whenever $K_n = H_1 \oplus H_2$, we must have $\omega_f(H_1) \geq x$ or $\omega_f(H_2) \geq y$. Then the *fractional Ramsey number* $r_f(x,y)$ is the least positive integer n for which $n \xrightarrow{f} (x,y)$. Note that we may meaningfully take x and y to be any real numbers greater than 1.

All graphs in this chapter are implicitly assumed to be finite. When we speak of $K_n = H_1 \oplus H_2$ as an edge 2-coloring, it is implicitly assumed that we are invoking the condition on $\oplus$ that $E(H_1) \cap E(H_2) = \emptyset$. However, a little thought reveals that the above Ramsey definitions with or without this condition are equivalent. We only note this because, in forthcoming constructions, it is sometimes convenient to take H_i's which are not edge-disjoint. However, if we find such H_i's with $\omega_f(H_i) < x_i$, removing edges from some H_i's to make them edge-disjoint does not cause this condition to be violated. So we adhere to the convention of $E(H_1) \cap E(H_2) = \emptyset$ only as is convenient.

3.1 The Value of $r_f(x, y)$

Unlike the ordinary Ramsey numbers, the exact value of $r_f(x, y)$ is known for any $x, y \geq 2$.

Theorem 3.1 *Let $x, y \in \mathbf{R}$ with $x, y > 1$. Express x and y as $x = k + \varepsilon$ and $y = l + \delta$, where $k, l \in \mathbf{N}$ and $0 < \varepsilon, \delta \leq 1$. Let $q = \min\{\lceil \varepsilon l \rceil, \lceil \delta k \rceil\}$. Then $r_f(x, y) = kl + q$.*

Proof. We first establish two basic facts about ϵ, δ, k, l and q as given above.

(i) Either $q = \lceil \varepsilon l \rceil \geq \varepsilon l$ or $q = \lceil \delta k \rceil \geq \delta k$, so that either $q/l \geq \varepsilon$ or $q/k \geq \delta$.

(ii) $q \leq \lceil \varepsilon l \rceil < \varepsilon l + 1$ and $q \leq \lceil \delta k \rceil < \delta k + 1$, so that $(q-1)/l < \varepsilon$ and $(q-1)/k < \delta$.

Now, $r_f(x, y)$ is clearly symmetric in x and y, and $r_f(x, y) = \lceil x \rceil$ is easily checked for $y \in (1, 2]$, so we restrict our attention to the case of x and y both greater than 2. Otherwise, take $x, y, k, \varepsilon, l, \delta$ and q as given above, and set $n = kl + q$. We establish that $r_f(x, y) = n$ in two steps: first, show that $n \xrightarrow{f} (x, y)$; then, construct a decomposition $K_{n-1} = H_1 \oplus H_2$ with $\omega_f(H_1) < x$ and $\omega_f(H_2) < y$ (which shows that $n - 1 \xrightarrow{f} (x, y)$ is false).

To show $n \xrightarrow{f} (x, y)$, let $K_n = H_1 \oplus H_2$ be *any* edge 2-coloring of K_n (so that $E(H_1) \cap E(H_2) = \emptyset$, $\omega(H_1) = \alpha(H_2)$ and vice versa). If $\omega(H_1) \geq k + 1$, then $\omega_f(H_1) \geq k + 1 \geq x$, and we're done. Since $\omega(H_2) \geq l + 1$ similarly implies $\omega_f(H_2) \geq y$, we may suppose that $\alpha(H_2) = \omega(H_1) \leq k$ and $\alpha(H_1) = \omega(H_2) \leq l$. Then by Lemma 1.3 and **(i)** above, either $q/l \geq \varepsilon$, in which case

$$\omega_f(H_1) \geq \frac{n}{\alpha(H_1)} \geq \frac{kl+q}{l} = k + \frac{q}{l} \geq k + \varepsilon,$$

or $q/k \geq \delta$, which gives

$$\omega_f(H_2) \geq \frac{n}{\alpha_(H_2)} \geq \frac{kl+q}{k} = l + \frac{q}{k} \geq l + \delta.$$

That one of these holds is exactly the statement $n \xrightarrow{f} (x, y)$.

To achieve $K_{n-1} = H_1 \oplus H_2$ with $\omega_f(H_1) < x$ and $\omega_f(H_2) < y$, we take $H_1 = C_{(n-1),l}$ and $H_2 = \overline{H_1}$. By Lemma 1.4 and **(ii)** above, this immediately gives

$$\omega_f(H_1) = \frac{n-1}{l} = \frac{kl+q-1}{l} = k + \frac{q-1}{l} < k + \varepsilon = x$$

and

$$\omega_f(H_2) = \frac{n-1}{\lfloor (n-1)/l \rfloor}.$$

We note that $(n-1)/l = k + \frac{q-1}{l} < k + \varepsilon$, and also that $(n-1)/l \geq k$ since q is at least 1. So $\lfloor (n-1)/l \rfloor = k$, and applying **(ii)** again gives

$$\omega_f(H_1) = \frac{n-1}{k} = \frac{kl+q-1}{k} = l + \frac{q-1}{k} < l + \delta = y.$$

These are exactly the properties we required of H_1 and H_2, so $n-1 \xrightarrow{f} (x, y)$ is false, and we're done. □

Corollary 3.2 *If $k \geq l \geq 2$ are integers, then $r_f(k, l) = kl - k$.* □

Note that, while $r(k, k)$ grows exponentially in k, $r_f(x, x)$ only grows quadratically in x. Two plots of $r_f(x, x)$ vs. x are shown in Figure 5.

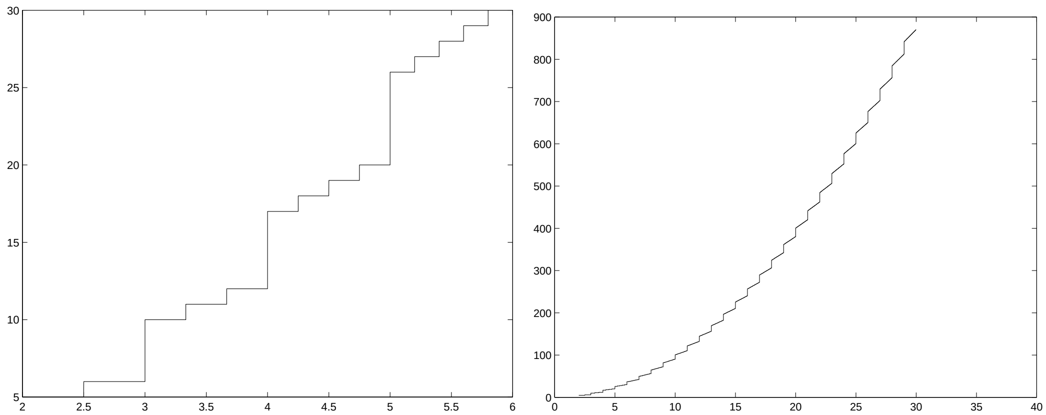

Figure 5: Two graphs of the function $y = r_f(x, x)$. Although this is a step function, its growth roughly conforms to a parabola. Large jumps occur at integer values of x, while size 1 jumps occur at even intervals between integers.

3.2 Multicolor Fractional Ramsey Numbers

We may extend the definition of Ramsey numbers by using more than two colors on the edges of K_n. We let $n \to (k_1, \ldots, k_p)$ mean that, whenever $K_n = H_1 \oplus \cdots \oplus H_p$, we must have $\omega(H_i) \geq k_i$ for some $i \in \{1, \ldots, p\}$. The Ramsey number $r(k_1, \ldots, k_p)$ is the least positive integer n for which this holds.

Similarly, we take $n \xrightarrow{f} (x_1, \ldots, x_p)$ to mean that, whenever $K_n = H_1 \oplus \cdots \oplus H_p$, we must have $\omega_f(H_i) \geq k_i$ for some $i \in \{1, \ldots, p\}$, and the fractional Ramsey number $r(x_1, \ldots, x_p)$ is the least positive integer for which this is true.

We may derive a recursive upper bound on r_f as follows:

Theorem 3.3 *Let* $x_1, \ldots, x_p \geq 2$. *Then*

$$r_f(x_1, \ldots, x_p) \leq \lceil (r_f(x_1, \ldots x_{p-1}) - 1)x_p \rceil.$$

Proof. Let $n = \lceil (r_f(x_1, \ldots, x_{p-1}) - 1)x_p \rceil$, and let $K_n = H_1 \oplus \cdots \oplus H_p$ be any p-

coloring of $E(K_n)$. Let $G = H_1 \oplus \cdots \oplus H_{p-1}$. If $\omega(G) \geq r_f(x_1, \ldots, x_{p-1})$, then G contains a complete subgraph large enough to guarantee that $\omega_f(H_i) \geq x_i$ for some $i \in \{1, \ldots, p-1\}$, and we're done. So we may suppose that $\omega(G) \leq r_f(x_1, \ldots, x_{p-1}) - 1$. Since $\alpha(H_p) = \omega(G)$ we have

$$\omega_f(H_p) \geq \frac{n}{\alpha(H_p)} \geq \frac{\lceil (r_f(x_1, \ldots, x_{p-1}) - 1)x_p \rceil}{r_f(x_1, \ldots, x_{p-1}) - 1} \geq x_p$$

as required. □

In fact, there are no known instances where this upper bound does not provide the correct value of r_f. For example, in the case of $p = 2$, the given bound reduces to

$$r_f(x, y) \leq \min \{ \lceil (\lceil x \rceil - 1)y \rceil , \lceil (\lceil y \rceil - 1)x \rceil \}$$

which is easily shown to be the value derived for $r_f(x, y)$ in Theorem 3.1. Of course, r_f is symmetric in its arguments, so in order for the expression in Theorem 3.3 to yield the best possible bound, we must at each recursive step choose the best x_i value to play the role of x_p in this expression. All of this implies the following conjecture.

Conjecture 3.4 *Let $x_1, \ldots, x_p \geq 2$. Then for some $i \in \{1, \ldots, p\}$,*

$$r_f(x_1, \ldots, x_p) = \lceil (r_f(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p) - 1)x_i \rceil . \square$$

In the case that the arguments are integers, the "correct" choice of x_i is simply the largest argument, so for integers $k_p \geq k_{p-1} \geq \cdots \geq k_1 \geq 2$ we have

$$r_f(k_1, \ldots, k_p) \leq \prod_{i=1}^{p} k_i - \prod_{i=2}^{p} - \cdots - k_{p-1}k_p - k_p .$$

On the other hand, it is easily shown[13] that

$$r_f(k_1, \ldots, k_p) > \prod_{i=1}^{p}(k_i - 1)$$

so our upper bound is at least on the right order of magnitude.

There are several other instances where our conjecture is known to be true, the most significant being the case where the arguments are all the *same* integer.

Theorem 3.5 *For integers* $k, p \geq 2$*,*

$$r_f(\underbrace{k, k, \ldots, k}_{p}) = k^p - k^{p-1} - \cdots - k.$$

We postpone the rather long proof of this theorem to mention a few other instances where Conjecture 3.4 is true. In the following three cases, the value of r_f is that given by Conjecture 3.4, and we give $K_{r_f-1} = H_1 \oplus H_2 \oplus H_3$, where each H_i is the indicated circulant graph, and I_i one of its independent sets of required size. We take $V(K_{r_f-1}) = V(H_i)$ to be $\{0, 1, \ldots, r_f - 2\}$, and note that $\omega_f(H_i) = (r_f - 1)/|I_i|$ (by Lemma 1.3).

- $r_f(x, y, z) = 20,\ \frac{19}{7} < x, y \leq 3,\ \frac{19}{5} < z \leq 4.$
  $H_1 = \langle\{2, 8\}\rangle_{19}, \quad I_1 = \{0, 3, 6, 9, 12, 15, 18\}, \quad \omega_f(H_1) = 19/7,$
  $H_2 = \langle\{4, 6\}\rangle_{19}, \quad I_2 = \{0, 1, 2, 3, 10, 11, 12\}, \quad \omega_f(H_2) = 19/7,$
  $H_3 = \langle\{1, 3, 5, 7, 9\}\rangle_{19}, \quad I_3 = \{0, 2, 4, 6, 8\}, \quad \omega_f(H_3) = 19/5.$

- $r_f(x, y, z) = 28,\ \frac{27}{10} < x \leq 3,\ \frac{27}{7} < y, z \leq 4.$
  $H_1 = \langle\{1, 4, 6\}\rangle_{27}, \quad I_1 = \{0, 2, 5, 7, 10, 12, 15, 17, 20, 22\},$
  $\omega_f(H_1) = 27/10,$

[13] See [6].

$H_2 = \langle\{2,3,5\}\rangle_{27}, \quad I_2 = \{0,1,7,8,14,15,21\}, \quad \omega_f(H_2) = 27/7,$

$H_3 = \langle\{7,8,\ldots,13\}\rangle_{27}, \quad I_3 = \{0,1,\ldots,6\}, \quad \omega_f(H_3) = 27/7.$

- $r_f(x,y,z) = 35$, $\frac{34}{14} < x \leq 3$, $\frac{34}{9} < y \leq 4$, $\frac{34}{7} < z \leq 5$.

  $H_1 = \langle\{14,15,16,17\}\rangle_{34}, \quad I_1 = \{0,1,\ldots,13\}, \quad \omega_f(H_1) = 34/14,$

  $H_2 = \langle\{4,5,9,10\}\rangle_{34}, \quad I_2 = \{0,1,2,8,14,15,21,22,28\},$

  $\omega_f(H_2) = 34/9,$

  $H_3 = \langle\{1,2,3,6,7,8,11,12,13\}\rangle_{34}, \quad I_3 = \{0,5,10,15,20,25,29\},$

  $\omega_f(H_3) = 34/7.$

There is a final general case where Conjecture 3.4 is true. Let $k_1, \ldots, k_p \geq 2$ be integers. Then if each of $\varepsilon_1, \ldots, \varepsilon_p > 0$ is sufficiently small, we have

$$r_f(k_1+\varepsilon_1, \ldots, k_p+\varepsilon_p) = 1 + \prod_{i=1}^{p} k_i.$$

That this is the value indicated by Conjecture 3.4 is easily checked. For the lower bound, consider $K_{k_1k_2} = K_{k_1}[K_{k_2}]$. Every edge from this graph comes either from K_{k_1} (between two copies of K_{k_2}) or K_{k_2} (within a copy of K_{k_2}). If we let H_1 contain all such K_{k_1} edges, and H_2 all such K_{k_2} edges, then $H_1 = K_{k_1}[\overline{K_{k_2}}]$ and $H_2 = \overline{K_{k_1}}[K_{k_2}]$. By Lemma 1.5, $\omega_f(H_1) = k_1$ and $\omega_f(H_2) = k_2$. More generally, if we take successive lexicographic products of the K_{k_i}'s, we get a complete graph on $k_1k_2\cdots k_p$ vertices. And if we let H_i consist of all edges which come from K_{k_i}, then

$$H_i = \overline{K_{k_1\cdots k_{i-1}}}[K_{k_i}[\overline{K_{k_{i+1}\cdots k_p}}]],$$

$$K_{k_1k_2\cdots k_p} = H_1 \oplus \cdots \oplus H_p, \quad \text{and}$$

$$\omega_f(H_i) = k_i < k_i + \varepsilon_i.$$

This shows that $k_1 \cdots k_p \xrightarrow{f} (k_1 + \varepsilon_1, \ldots, k_p + \varepsilon_p)$ is false.

Finally note that, while choosing the largest x_i gives the best bound in the case that all arguments are integers, this is not in general the case. To calculate $r_f(3.1, 3.1, 4.9)$, first note that $r_f(3.1, 3.1) = 10$ and $r_f(3.1, 4.9) = 13$. Then

$$\begin{aligned} r_f(3.1, 3.1, 4.9) &\leq \lceil (r_f(3.1, 3.1) - 1)4.9 \rceil = 45, \qquad \text{but} \\ r_f(3.1, 3.1, 4.9) &\leq \lceil (r_f(3.1, 4.9) - 1)3.1 \rceil = 38. \end{aligned}$$

The second bound is clearly better, even though the largest x_i was not used as the recursion point.

We now return to...

Proof of Theorem 3.5. We restrict our attention to the case of $k \geq 3$, as the $k = 2$ case is trivially true. And since we've already proved Theorem 3.3, all that remains to be shown is that $r_f(k, \ldots, k) > k^p - k^{p-1} - \cdots - k - 1$. As in the 2-color proof, this is accomplished by constructing $K_{r_f(k,\ldots,k)-1} = H_1 \oplus \cdots \oplus H_p$ (an edge p-coloring of the complete graph) such that $\omega_f(H_i) < k$ for all i. And as before, this is accomplished via the use of circulant graphs. We use the more general form $\langle S \rangle_n$, and will generally suppress the n subscript, as its value will be clear in context.

In the following, we will hold k fixed and let p vary. Let $n_p = k^p - k^{p-1} - \cdots - k - 1$, the order of the complete graph whose edges we are p-coloring. Notice that $n_{p+1} = kn_p - 1$, and since $n_1 = k - 1$, we set $n_0 = 1$ for consistency. Since the H_i's will be chosen to be circulant graphs, they will be vertex transitive, and thus $\omega_f(H_i) = \frac{n_p}{\alpha(H_i)}$ by Lemma 1.3. Since we want $\omega_f(H_i) < k$, we let α_p be the desired value of $\alpha(H_i)$ in the p-color case.

Specifically,

$$\begin{aligned}\alpha_p &= \min\{\alpha \in \mathbf{N} : \frac{n_p}{\alpha} < k\} \\ &= \min\{\alpha \in \mathbf{N} : \alpha > \frac{kn_{p-1} - 1}{k} = n_{p-1} - \frac{1}{k}\} \\ &= n_{p-1} \ ,\end{aligned}$$

with $\alpha_1 := 1$. (Henceforth, n_{p-1} and α_p are used interchangeably).

Let $d_p = \lfloor \frac{n_p}{2} \rfloor$, so that $K_{n_p} = \langle \{1, \ldots, d_p\} \rangle_{n_p}$. Then in order to have $K_{n_p} = H_1 \oplus \cdots \oplus H_p$ where each H_i is a circulant graph, we require that the union of the connection distance sets of all the H_i's be exactly $\{1, \ldots, d_p\}$. Note that in the following construction, the H_i's will not be edge-disjoint (the connection distance sets will not be disjoint). Let us superscript each H_i with its corresponding p, and then let S_i^p be the set of connection distances defining H_i^p, i.e., $H_i^p = \langle S_i^p \rangle$. For a set of integers S and an integer r, we define $S + r := \{s + r : s \in S\}$. Similarly $r - S := \{r - s : s \in S\}$.

We begin our construction by defining

$$\begin{aligned}S_i &:= \{n_{i-1}, \ldots, n_i - n_{i-1}\} \\ h_i &:= (k-1)n_i - 1 = n_{i+1} - n_i = k^{i+1} - 2k^i\end{aligned}$$

and then

$$\begin{aligned}S_i^p &:= \{s \le d_p : s \bmod h_i \in S_i\} \\ &\subset S_i \cup (S_i + h_i) \cup (S_i + 2h_i) \cup \cdots\end{aligned}$$

Notice that, since $k \geq 3$, for $p \geq 1$ we have that $kn_{p-1} - 1 \geq 2n_{p-1}$. Then $n_p/2 \geq n_{p-1}$, and we have

$$n_p - n_{p-1} \ \geq \ \frac{n_p}{2}$$

$$\geq \lfloor \frac{n_p}{2} \rfloor = d_p$$
$$\geq n_{p-1} .$$

Therefore $\{n_{p-1}\} \subseteq \{n_{p-1}, \ldots, d_p\} \subseteq \{n_{p-1}, \ldots, n_p - n_{p-1}\}$, and $S_p^p = \{n_{p-1}, \ldots, d_p\}$. And since $n_p - n_{p-1} \leq n_p = n_{p-1} + h_{p-1}$, it follows that $S_{p-1}^p = S_{p-1}$. Finally, note that $S_i^p \subset S_i^{p+1}$.

Example

To help illustrate the proof, we consider the case $k = 4$. Here we present the items we have defined thus far for $p = 1, 2, 3, 4$.

$$\begin{array}{llll} n_1 = 3 & n_2 = 11 & n_3 = 43 & n_4 = 171 \\ d_1 = 1 & d_2 = 5 & d_3 = 21 & d_4 = 85 \\ h_1 = 8 & h_2 = 32 & h_3 = 128 & \\ \alpha_1 = 1 & \alpha_2 = 3 & \alpha_3 = 11 & \alpha_4 = 43 \end{array}$$

$$\begin{aligned} S_1 &= \{1, 2\} \\ S_2 &= \{3, 4, 5, 6, 7, 8\} \\ S_3 &= \{11, 12, \ldots, 32\} \\ S_4 &= \{43, 44, \ldots, 128\} \end{aligned}$$

$$S_1^1 = \{1\}$$

$$\begin{aligned} S_1^2 &= \{1, 2\} \\ S_2^2 &= \{3, 4, 5\} \end{aligned}$$

$$\begin{aligned} S_1^3 &= \{1, 2, 9, 10, 17, 18\} \\ S_2^3 &= \{3, 4, 5, 6, 7, 8\} \\ S_3^3 &= \{11, 12, \ldots, 21\} \end{aligned}$$

$$\begin{aligned} S_1^4 &= \{1, 2, 9, 10, 17, 18, 25, 26, 33, 34, 41, 42, \\ &\qquad 49, 50, 57, 58, 65, 66, 73, 74, 81, 82\} \\ S_2^4 &= \{3, \ldots, 8, 35, \ldots, 40, 67, \ldots, 72\} \\ S_3^4 &= \{11, 12, \ldots, 32\} \\ S_4^4 &= \{43, 44, \ldots, 85\} \end{aligned}$$

We return to the example of $k = 4$ several times to further clarify our constructions.

We now use induction to verify that this construction actually does cover all connection distances $1, \ldots, d_p$, (i.e., that $S_1^p \cup \cdots \cup S_p^p = \{1, \ldots, d_p\}$, and therefore $K_{n_p} = H_1^p \oplus \cdots \oplus H_p^p$). We have $S_1^1 = \{1, \ldots, d_p\}$ as a base case, so we now suppose that $S_1^p \cup \cdots \cup S_p^p = \{1, \ldots, d_p\}$. Since $S_{p+1}^{p+1} = \{n_p, \ldots, d_{p+1}\}$, we need only show that $\{1, \ldots, n_p - 1\} \subset (S_1^{p+1} \cup \cdots \cup S_p^{p+1})$.

Take $i < p$. Since $h_i = k^{i+1} - 2k^i$, we compute that

$$\begin{aligned} n_p &= k^p - k^{p-1} - k^{p-2} - \cdots - k - 1 \\ &= (k^{i+1} - 2k^i)(k^{p-i-1} + \cdots + 1) + k^i - k^{i-1} - \cdots - 1 \\ &= h_i(k^{p-i-1} + \cdots + 1) + n_i, \end{aligned}$$

so $n_p \bmod h_i = n_i$. Next, if $x \in S_i^p$, then $n_p - x \in n_p - S_i^p$, and

$$\begin{aligned} (n_p - x) \bmod h_i \in [(n_p - S_i^p) \bmod h_i] &\subseteq [(n_p - S_i) \bmod h_i] \\ &= (n_i - S_i) \bmod h_i \\ &= (n_i - \{n_{i-1}, \ldots, n_i - n_{i-1}\}) \bmod h_i \\ &= \{n_{i-1}, \ldots, n_i - n_{i-1}\} \\ &= S_i. \end{aligned}$$

We also know that $n_p - x < n_p \leq d_{p+1}$, so $n_p - x \in S_i^{p+1}$. Since $S_1^p, \ldots, S_p^p$ cover $\{1, \ldots, d_p\}$ (our induction hypothesis), it then follows that $S_1^{p+1}, \ldots, S_p^{p+1}$ cover $\{n_p - d_p, \ldots, n_p - 1\}$ as well as $\{1, \ldots, d_p\}$ (recall that $S_i^p \subset S_i^{p+1}$). But since $d_p = \lfloor \frac{n_p}{2} \rfloor$, we have that $\{1, \ldots, d_p\} \cup \{n_p - d_p, \ldots, n_p - 1\} = \{1, \ldots, n_p - 1\}$, all of which is covered by $S_1^{p+1}, \ldots, S_p^{p+1}$, as desired. Thus all connection distances are covered, and it follows that $K_{n_p} = H_1^p \oplus \cdots \oplus H_p^p$.

We must now show that $\alpha(H_i^p) \geq \alpha_p$, so that $\omega_f(H_i^p) < k$. We accomplish this by

(i) defining a set $I_i^p \subseteq \{0, 1, \ldots, n_p - 1\}$,

(ii) showing that I_i^p is an independent set in H_i^p, and

(iii) showing that $|I_i^p| \geq \alpha_p$.

(i) Constructing I_i^p

We first define the sets

$$\begin{aligned} B_i &= \{0, 1, \ldots, \alpha_i - 1\}, \\ C_i &= \{0, 1, \ldots, \alpha_i - 2\}, \text{ and} \\ D_i &= B_i \cup (B_i + n_i) \cup \cdots \cup [B_i + (k-3)n_i] \cup [C_i + (k-2)n_i]. \end{aligned}$$

We then define our independent set I_i^p to be

$$I_i^p = D_i \cup (D_i + h_i) \cup \cdots \cup (D_i + r_{i,p}h_i) \cup [B_i + (r_{i,p} + 1)h_i],$$

where

$$r_{i,p} = \max\{t : \alpha_i - 1 + (t+1)h_i < n_p\}.$$

That is to say, $r_{i,p}$ is the largest possible value so that no element of I_i^p exceeds n_p. Specifically, $r_{i,p} = k^{p-i-1} + k^{p-i-2} + \cdots + k$. To see this, add n_i (which is much smaller than the span of a D_i, but larger than the span of B_i) to the leading 0 in the last B_i in I_i^p. We get

$$\begin{aligned} (r_{i,p} + 1)h_i + n_i &= (k^{p-i-1} + \cdots + k + 1)(k-1)n_i - (r_{i,p} + 1) + n_i \\ &= (k^{p-i} - 1)n_i + n_i - (r_{i,p} + 1) \\ &= (k^p - k^{p-1} - \cdots - k^{p-i}) - (k^{p-i-1} + \cdots + k + 1) \\ &= n_p, \end{aligned}$$

so the given value of $r_{i,p}$ implies I_i^p is a subset of $\{0,\ldots,n_p-1\}$. In the case of $i=p-1$, this gives $r_{i,p}=0$, and we just have $I_{p-1}^p = D_{p-1}\cup(B_{p-1}+h_{p-1})$. Finally, we set $I_p^p = B_p = \{0,\ldots,\alpha_p-1\}$. This completes our construction.

Example

Here, we list independent sets I_i^p and their components in the case of $k=4$ and $p=1,2,3,4$. A better illustration of these patterns for $i=2$ may be found in Figure 6.

$$
\begin{aligned}
B_1 &= \{0\}\\
C_1 &= \emptyset\\
D_1 &= B_1\cup(B_1+3)\cup(C_1+6)=\{0,3\}\\
\\
B_2 &= \{0,1,2\}\\
C_2 &= \{0,1\}\\
D_2 &= B_2\cup(B_2+11)\cup(C_2+22)=\{0,1,2,11,12,13,22,23\}\\
\\
B_3 &= \{0,1,\ldots,10\}\\
C_3 &= \{0,1,\ldots,9\}\\
D_3 &= B_3\cup(B_3+43)\cup(C_3+86)=\{0,1,\ldots,10,43,44,\ldots,53,86,87,\ldots,95\}\\
\\
h_1 &= 8\\
h_2 &= 32\\
h_3 &= 128\\
\\
I_1^1 &= \{0\}\\
\\
I_1^2 &= \{0,3,8\}\\
I_2^2 &= \{0,1,2\}\\
\\
I_1^3 &= \{0,3,8,11,16,19,24,27,32,35,40\}\\
I_2^3 &= \{0,1,2,11,12,13,22,23,32,33,34\} \qquad \text{See Figure 6}\\
I_3^3 &= \{0,1,2,3,4,5,6,7,8,9,10\}\\
\\
I_1^4 &= \{0,3,8,11,16,19,24,27,32,35,40,43,48,51,56,59,\\
&\quad 64,67,72,75,80,83,88,91,96,99,104,107,112,115,\\
&\quad 120,123,128,131,136,139,144,147,152,155,160,163,
\end{aligned}
$$

$$
\begin{aligned}
& \quad 168\} \\
I_2^4 &= \{0, 1, 2, 11, 12, 13, 22, 23, \\
& \quad 32, 33, 34, 43, 44, 45, 54, 55, \\
& \quad 64, 65, 66, 75, 76, 77, 86, 87, \\
& \quad 96, 97, 98, 107, 108, 109, 118, 119, \\
& \quad 128, 129, 130, 139, 140, 141, 150, 151, \\
& \quad 160, 161, 162\} \qquad \text{See Figure 6} \\
I_3^4 &= \{0, \ldots, 10, 43, \ldots, 53, 86, \ldots, 95, 128, \ldots, 138\} \\
I_4^4 &= \{0, \ldots, 42\}
\end{aligned}
$$

(ii) I_i^p is an independent set of H_i^p

Given a subset I of the vertices of a circulant graph $\langle S \rangle$, I is an independent set of $\langle S \rangle$ if, for any two $x, y \in I$, the distance between them along their shorter around the circle (i.e. their "connection distance") is not in the set S of connection distances of the graph. We check that this condition is satisfied in each of three cases, dependent on the value of i.

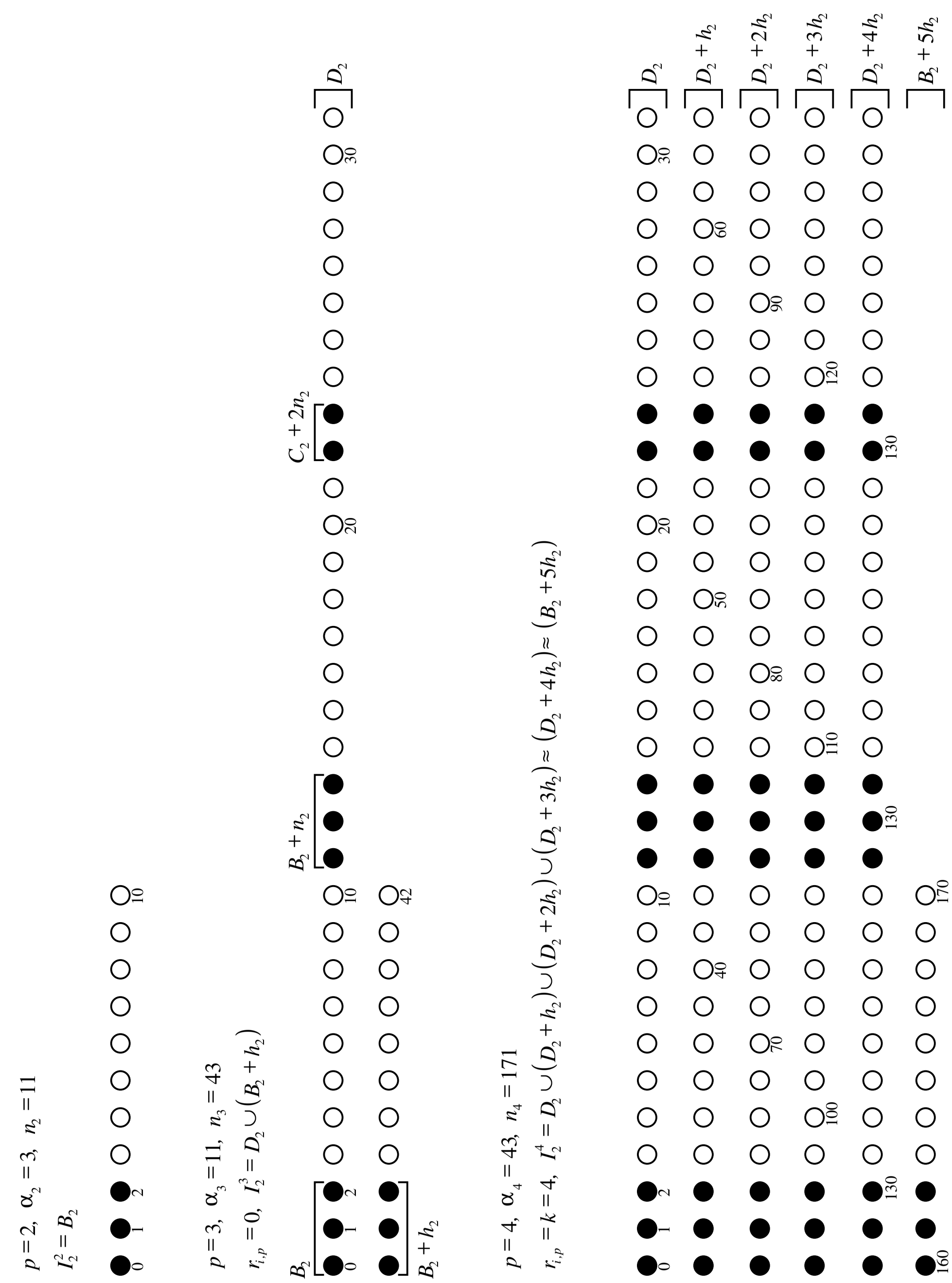

Figure 6: The desired independent sets I_i^p of H_i^p for $i = 2$ and $p = 2, 3, 4$. Note that, while the vertices are drawn here in rows for clarity, they should be imagined to be laid out around the perimeter of a circle.

(ii.a) I_p^p is an independent set of H_p^p

Recall that $S_p^p = \{\alpha_p, \ldots, d_p\}$ and $I_p^p = \{0, \ldots, \alpha_p - 1\}$ (where $\alpha_p - 1 < n_p/2$). The elements of I_p^p are therefore consecutive along the circle, and none are more than $\alpha_p - 1$ steps apart. Since the smallest connection distance is α_p, no pair of vertices in I_p^p can be adjacent.

(ii.b) I_{p-1}^p is an independent set of H_{p-1}^p

Recall that $S_{p-1}^p = S_{p-1} = \{\alpha_{p-1}, \ldots, \alpha_p - \alpha_{p-1}\}$, and that I_{p-1}^p contains exactly one copy of D_{p-1}. To show that I_{p-1}^p is an independent set of H_{p-1}^p, we consider all possible connection distances within I_{p-1}^p, and observe that none of them is found in S_{p-1}^p.

Note that I_{p-1}^p is just the union of translated copies of B_{p-1} and C_{p-1}. Any pair of vertices in the same copy of B or C are at most $\alpha_{p-1} - 1$ steps apart. Since the smallest connection distance (element of S_{p-1}^p) is α_{p-1}, these vertices cannot be adjacent, so we turn our attention to vertices in different copies of B and C. The *smallest* possible distance between such a pair of vertices occurs between the last vertex u in one block (B or C) and the first vertex v of the next block. We now prove that, in all possible cases, this smallest possible distance is larger than $\alpha_p - \alpha_{p-1}$ (the largest value in S_{p-1}^p).

- *u is last vertex in $B_{p-1} + jn_{p-1}$ and v is the first vertex in $B_{p-1} + (j+1)n_{p-1}$ with* $0 \leq j < k - 3$.

  In this case we have $u = (\alpha_{p-1} - 1) + jn_{p-1}$ and $v = 0 + (j+1)n_{p-1}$, so the distance between them is $v - u = n_{p-1} - (\alpha_{p-1} - 1) = \alpha_p - \alpha_{p-1} + 1$, as desired.

- *u is the last vertex in $B_{p-1} + (k-3)n_{p-1}$ and v is the first vertex in $C_{p-1} + (k-2)n_{p-1}$.*

  As above, the distance between u and v is $n_{p-1} - (\alpha_{p-1} - 1) = \alpha_p - \alpha_{p-1} + 1$.

- *u is the last vertex in $C_{p-1} + (k-2)n_{p-1}$ and v is the first vertex in $B_{p-1} + h_{p-1}$.*

  Here, $u = (\alpha_{p-1} - 2) + (k-2)n_{p-1}$ and $v = 0 + h_{p-1} = (k-1)n_{p-1} - 1$. The distance between them is $\alpha_p - \alpha_{p-1} + 1$, as desired.

- *u is the last vertex in $B_{p-1} + h_{p-1}$ and v is the first vertex in B_{p-1}.*

  In this case $u = h_{p-1} + \alpha_{p-1} - 1$ and $v = 0$. Their distance is $n_p - (h_{p-1} + \alpha_{p-1} - 1) = \alpha_p - \alpha_{p-1} + 1$, as desired.

Recapping... any two vertices in the same B or C block of I^p_{p-1} have connection distance less that α_{p-1}, and so less than any value in $S^p_{p-1} = S_{p-1}$; any two vertices in different blocks of I^p_{p-1} have connection distance greater than $\alpha_p - \alpha_{p-1}$, and so greater than any value in S^p_{p-1}. (Keep these facts in mind for the following section.) Thus I^p_{p-1} is an independent set of H_{p-1}.

(ii.c) I^p_i is an independent set of H^p_i for $1 \leq i < p-1$

Recall that, like I^i_{i-1}, I^p_i is a collection of translated B_i's and C_i's. So as before, the distance between any two vertices in distinct blocks of I^p_i is at least $\alpha_{i+1} - \alpha_i + 1$. All such B_i's and C_i's (except for the final copy of B_i) are contained in translated copies of D_i, which are spaced at intervals of h_i. If we consider some $x \in I^p_i$, and we add or subtract h_i to it, we are essentially moving it to the same position within the next or the previous copy of D_i (so long as this movement does not push, or "scroll", x across the $n_p - 1$ to 0 gap).

Now, let $x, y \in I^p_i$ and let d denote the distance between them along the shorter arc of the circle. Without loss of generality, assume that $0 \leq y < x$. Note that $d = x - y$ or $d = n_p - (x - y)$, depending on whether or not the shorter arc covers the $n_p - 1$ to 0

gap. We must show that $d \notin S_i^p$, and since $S_i^p \bmod h_i = S_i$, it is enough to show that $d' := d \bmod h_i$ is not in S_i.

We first consider the case where $d = x - y$ (so the shorter arc between y and x does not cover the $n_p - 1$ to 0 gap). We may subtract multiples of h_i from x to give $x' \in I_i^p$ with the property that $0 \leq y \leq x' < y + h_i$. The distance between x' and y is $d' = d \bmod h_i$. Now either x' and y are in the same copy of D_i or in consecutive copies of D_i. In either case, they are either in the same B_i or C_i block (so, as before, $d' < \alpha_{i-1}$) or in different B_i or C_i blocks (so $d' > \alpha_{i+1} - \alpha_i$). Therefore $d' \notin S_i$, and we conclude that x and y are not adjacent in H_i^p.

We now turn our attention to the more difficult case of $d = n_p - (x - y)$, where the shorter arc between x and y covers the $n_p - 1$ to 0 gap. Here, we may subtract a multiple of h_i from y and add a multiple of h_i to x to yield vertices x' and y' where y' is in the first copy of D_i in I_i^p and x' is either in the last copy of D_i (i.e., $x' \in D_i + r_{i,p}h_i$) or in the terminal B_i (i.e., $x' \in B_i + (r_{i,p} + 1)h_i$). Setting $d'' = n_p - (x' - y')$, we consider three possible cases.

Case (a): $d'' < h_i$.

In this case, $d'' = d' := d \bmod h_i$. Now, x' and y' are at least as far apart as the last and first elements in I_i^p, which, as in the similar case from (ii.b), are at a distance of $\alpha_{i+1} - \alpha_i + 1$. This is too large to be in S_i, so $d' \notin S_i$ and $d \notin S_i^p$.

Case (b): $x' \in D_i + r_{i,p}h_i$, $d'' \geq h_i$.

In this case we add h_i to x' to get x'', but in moving x' across zero, it crosses an extra B_i. This extra distance is subtracted from x''s relative position within its D_i. This distance is exactly n_i, which is the space between B_i's within D_i. Therefore x'''s position in the first D_i is exactly one B_i block back[14] from the position of x' in the last D_i. We still have x'' "behind" y' since d'' was at least h_i, but now both are contained in the first D_i at a distance of $y' - x'' = d' = d \bmod h_i$. Now d' is a distance between two elements of D_i, and we have seen before that $d' \notin S_i$, so that $d \notin S_i^p$.

Case (c): $x' \in B_i + (r_{i,p}+1)h_i\ ,\ \ d'' \geq h_i$.

Write $x' = z + (r_{i,p}+1)h_i$ where $z \in B_i$, i.e., z is x''s relative position in the terminal B_i. Adding h_i to this mod n_p gives

$$\begin{aligned} z + (r_{i,p}+1)h_i + h_i - n_p &= z + h_i - n_i + [(r_{i,p}+1)h_i + n_i - n_p] \\ &= z + h_i - n_i \\ &= (z-1) + (k-2)n_i. \end{aligned}$$

So after being moved from x' to $x'' = (x' + h_i) \bmod n_p$, we have $x'' \in [(k-2)n_i + \{-1, 0, \ldots, \alpha_i - 2\}]$. Notice that this set contains the C_i in the first (untranslated) D_i, and that y must be in this C_i (otherwise x' and y' would have been closer than h_i, putting us in Case (a)). So x'' and y' are both in this set $[(k-2)n_i + \{-1, 0, \ldots, \alpha_i - 2\}]$, and the distance between any two elements in it is at most $\alpha_i - 1 < s$ for all $s \in S_i$, so as before, $d \notin S_i^p$.

Thus in all cases, $d \notin S_i^p$, i.e., no connection distance between points in I_i^p is found in S_i^p, so I_i^p must be an independent set of $\langle S_i^p \rangle = H_i^p$.

(iii) $|I_i^p| = \alpha_p$

[14]We do not allow x' to be in the first B_i block of the last D_i;. If it were, a shift forward by h_i would move it into the terminal B_i in I_i^p. This situation is covered separately in Case (c) below.

$$
\begin{aligned}
|I_i^p| &= (r_{i,p}+1)|D_i| + |B_i| \\
&= (r_{i,p}+1)\left[(k-2)|B_i| + |C_i|\right] + \alpha_i \\
&= (r_{i,p}+1)[(k-2)\alpha_i + \alpha_i - 1] + \alpha_i \\
&= (r_{i,p}+1)[(k\alpha_i - 1) - \alpha_i] + \alpha_i \\
&= (r_{i,p}+1)\alpha_{i+1} - r_{i,p}\alpha_i \\
&= (r_{i,p}+1)(k^i - k^{i-1} - \cdots - 1) - r_{i,p}(k^{i-1} - k^{i-2} - \cdots - 1) \\
&= (r_{i,p}+1)k^i - (2r_{i,p}+1)k^{i-1} - k^{i-2} - \cdots - 1 \\
&= [(r_{i,p}+1)k - (2r_{i,p}+1)]k^{i-1} - k^{i-2} - \cdots - 1 \\
&= [(k^{p-i} + \cdots + k) - (2k^{p-i-1} + \cdots + 2k + 1)]k^{i-1} - k^{i-2} - \cdots - 1 \\
&= [k^{p-i} - k^{p-i-1} - \cdots - k - 1]k^{i-1} - k^{i-2} - \cdots - 1 \\
&= k^{p-1} - k^{p-2} - \cdots - k^{i-1} - k^{i-2} - \cdots - 1 \\
&= \alpha_p
\end{aligned}
$$

so I_i^p is of the desired size, even for $i = p-1$ ($r = 0$). Thus we have shown that H_i^p has an independent set of size α_p, which is exactly what we needed in order to verify that $\omega_f(H_i^p) < k$, and so we are done. □

3.3 A Relaxation of Fractional Ramsey Numbers

When we take $n \to (k,l)$ to mean that $K_n = H_1 \oplus H_2$ implies $\omega(H_1) \geq k$ or $\omega(H_2) \geq l$, we could equally well write this as "If $G = H_1 \oplus H_2$ and $\omega(G) \geq n$, then $\omega(H_1) \geq k$ or $\omega(H_2) \geq l$." We can now fractionalize the entirety of this statement, and take $z \xrightarrow{*} (x,y)$

to mean "If $G = H_1 \oplus H_2$ and $\omega_f(G) \geq z$, then $\omega_f(H_1) \geq x$ or $\omega_f(H_2) \geq y$." We then define $r^*(x, y)$ to be the infimum of all z for which this statement holds. We may also define the multicolor version of this, where $z \stackrel{*}{\longrightarrow} (x_1, \ldots, x_p)$ means that, if $G = H_1 \oplus \cdots \oplus H_p$ and $\omega_f(G) \geq z$, then $\omega_f(H_i) \geq x_i$ for some i. Likewise, $r^*(x_1, \ldots, x_p)$ is the infimum of all z for which $z \stackrel{*}{\longrightarrow} (x_1, \ldots, x_p)$ is true.

To achieve our result, we need the following lemma.

Lemma 3.6 *If $G = H_1 \oplus H_2$, then $\omega_f(G) \leq \omega_f(H_1)\omega_f(H_2)$.*

Proof. We prove this statement in the form $\chi_f(G) \leq \chi_f(H_1)\chi_f(H_2)$. Let a_1, a_2, b_1, b_2 be positive integers such that $\chi_f(H_i) = a_i/b_i = \chi_{b_i}(H_i)/b_i$ for $i = 1, 2$ (the proof of Lemma 1.1 guarantees that such integers exist). Let c_i be a proper b_i-fold coloring of H_i using a set of a_i colors, where $c_i(v)$ is the set of b_i colors assigned to $v \in V(G)$, and if $uv \in E(H_i)$ then $c_i(u) \cap c_i(v) = \emptyset$.

We may now construct a b_1b_2-fold coloring of G using a set of a_1a_2 colors: assign to the vertex v the set of colors $c_1(v) \times c_2(v)$. Now, if $uv \in E(G)$, then $uv \in E(H_1)$ or $uv \in E(H_2)$, which in turn implies that either $c_1(u) \cap c_1(v) = \emptyset$ or $c_2(u) \cap c_2(v) = \emptyset$. This guarantees that $c_1(u) \times c_2(u)$ and $c_1(v) \times c_2(v)$ are disjoint, and so we have a proper b_1b_2-fold coloring of G. This shows that $\chi_{b_1b_2}(G) \leq a_1a_2$, and so

$$\chi_f(G) = \inf_b \frac{\chi_b(G)}{b} \leq \frac{\chi_{b_1b_2}(G)}{b_1b_2} \leq \frac{a_1a_2}{b_1b_2} = \chi_f(H_1)\chi_f(H_2). \square$$

Theorem 3.7 *For real numbers $x_1, \ldots, x_p > 2$, we have $r^*(x_1, \ldots, x_p) = x_1x_2 \cdots x_p$.*

Proof. For any G with $\omega_f(G) \geq x_1 \cdots x_p$ and any decomposition $G = H_1 \oplus \cdots \oplus H_p$, if we suppose that $\omega_f(H_i) < x_i$ for all i, then Lemma 3.6 (applied repeatedly) implies that $\omega_f(G) < x_1 \cdots x_p$. This is a contradiction, and so we must have $\omega_f(H_i) \geq x_i$ for some i, as desired. Therefore $x_1 \cdots x_p \xrightarrow{*} (x_1, \ldots, x_p)$, and $r^*(x_1, \ldots, x_p) \leq x_1 \cdots x_p$.

We prove the lower bound by inducting on p.

BASE • $p = 1$:

That $r^*(x) = x$ is immediate.

INDUCTION HYPOTHESIS • Suppose that $r^*(x_1, \ldots, x_{p-1}) = x_1 \cdots x_{p-1}$:

Take any $z < x_1 \cdots x_p$. We must find a graph G and a decomposition $G = H_1 \oplus \cdots \oplus H_p$ where $\omega_f(G) \geq z$ and $\omega_f(H_i) < x_i$ for all i. To start, we choose rational q_1 and q_2 such that

$$2 < q_1 < x_1 \cdots x_{p-1}, \qquad 2 < q_2 < x_p \quad \text{and} \quad q_1 q_2 \geq z\ .$$

By our induction hypothesis, there is a graph with decomposition $G_{p-1} = H'_1 \oplus \cdots \oplus H'_{p-1}$ such that $\omega_f(G_{p-1}) \geq q_1$ but $\omega_f(H'_i) < x_i$ for each $i = 1, \ldots, p-1$. Recalling that $\omega_f(C_{(q_2)}) = q_2$, we take $G = C_{(q_2)}[G_{p-1}]$ (wherein each vertex of $C_{(q_2)}$ is replaced with a copy of G_{p-1}), so we have $\omega_f(G) \geq q_1 q_2 \geq z$ by Lemma 1.5. Let $C_{(q_2)}$ have m vertices, and G_{p-1} have n vertices. We may partition the edges of G into the set of edges *within* copies of G_{p-1}, and the set of edges *between* copies of G_{p-1}. The former yields a graph of the form

$$\overline{K_m}[G_{p-1}] = \overline{K_m}[H'_1 \oplus \cdots \oplus H'_{p-1}] = \overline{K_m}[H'_1] \oplus \cdots \oplus \overline{K_m}[H'_{p-1}],$$

while the later is $C_{(q_2)}[\overline{K_n}]$. So if we set $H_i = \overline{K_m}[H'_i]$ for $i = 1, \ldots, p-1$, and $H_p =$

$C_{(q_2)}[\overline{K_n}]$, then $G = H_1 \oplus \cdots \oplus H_p$. Further,

$$\omega_f(H_i) = \omega_f(\overline{K_m})\omega_f(H_i') = \omega_f(H_i') < x_i \quad \text{for } i = 1, \ldots, p-1$$

$$\omega_f(H_p) = \omega_f(C_{(q_2)})\omega_f(\overline{K_n}) = \omega_f(C_{(q_2)}) = q_2 < x_p$$

This construction shows that $z \xrightarrow{*} (x_1, \ldots, x_p)$ is false, and so $r^*(x_1, \ldots, x_p) \geq x_1 x_2 \cdots x_p$. Since this relation is implied by our induction hypothesis, we are done. □

3.4 b-Ramsey Numbers

We may define the *b-Ramsey number* of a graph by replacing ω in the Ramsey definition with ω_b' instead of ω_f. Recall from Section 1.1 that

$$\omega_b'(G) = \max \mathbf{1}\cdot\mathbf{y} \text{ s.t. } M'\mathbf{y} \leq \mathbf{1}, \quad \mathbf{y} \in \{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}\}^n \ .$$

A b-clique of G is a function $g_b : V(G) \to \{0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b}{b}\}$ such that $\sum_{v \in I} g_b(v) \leq 1$ for all $I \in \mathcal{I}$. The weight, or value, of this b-clique is $w(g_b) = \sum_{v \in V(G)} g_b(v)$, and $\omega_b'(G)$ is the minimum[15] value of $w(g_b)$ taken over all b-cliques of G.

We let $n \xrightarrow{b} (x, y)$ stand for the statement "If $K_n = H_1 \oplus H_2$, then $\omega_b'(H_1) \geq x$ or $\omega_b'(H_2) \geq y$." Then the b-Ramsey number $r_b(x, y)$ is the least positive integer n for which this statement is true.

Because $\omega_f(G) \geq \omega_b'(G) \geq \omega(G)$ for any positive integer b, it immediately follows that $r_f(x, y) \leq r_b(x, y) \leq r(x, y)$. In general, because $r_b(x, y)$ involves a discrete optimization invariant similar to $r(k, l)$, we expect computation of these values to be difficult (as opposed to the relative ease of calculating $r_f(x, y)$). We do, however, have two principle results

[15]Since we are not discussing infinite graphs, we do mean "minimum" and not "infimum".

regarding the limiting behavior of r_b. The first of these tells us that $r_2(k,k)$, like $r(k,k)$, grows exponentially in k, although the bound achieved on this growth rate is considerably smaller.

Lemma 3.8 *The edge set of K_n contains at least $\frac{1}{6}(n-3)(n-4)$ edge-disjoint triangles.*

Proof. It is a long-known result[16] that there is a Steiner triple system on n elements iff $n \equiv 1$ or $3 \bmod 6$. In other words, for sets of size $n \equiv 1$ or $3 \bmod 6$, we may find a collection T of size 3 subsets such that every pair of elements appears in exactly one member of T. This is equivalent to saying that T partitions $E(K_n)$ into triangles. Such a partition necessarily has $n(n-1)/6$ triangles. Even if $n \equiv 0 \bmod 6$, we may still partition the edges of a K_{n-3} subgraph into $(n-3)(n-4)/6$ triangles, giving the desired result. □

Note that, while the above value is not always best possible, it still approaches $n^2/6$ asymptotically, which clearly is the best possible limit.

Theorem 3.9 *If positive integers n and k satisfy*

$$2\binom{n}{k}\left(\frac{7}{8}\right)^{(k-3)(k-4)/6} < 1,$$

then $r_2(k,k) > n$.

Proof. We start by defining

$$\bar{\omega}_2(G) = \max \mathbf{1}\cdot\mathbf{y} \text{ s.t. } M'\mathbf{y} \le \mathbf{1}, \quad \mathbf{y} \in \{0, \frac{1}{2}\}^n ,$$

which is identical to $\omega_b'(G)$, except that we are only allowed to assign weights 0 or 1/2, and not 1, to vertices. We refer to feasible solutions to this program as *half-cliques*. Note

[16] First shown by Kirkman around 1850; see [1].

that, in any such half-clique, the set of vertices receiving weight 1/2 must be triangle-free, and in fact, $\bar{\omega}_2(G)$ is 1/2 times the number of vertices in the largest triangle-free induced subgraph of G. Now, the set of positively weighted vertices in any 2-clique must also be triangle free, but here we are using weights 1/2 and 1. Multiplying $\bar{\omega}_2(G)$ by 2 is equivalent to assigning weight 1 to every vertex in the *largest* triangle-free subgraph, so we must have $\omega_2'(G) \leq 2\bar{\omega}_2(G)$.

We now employ a probabilistic technique similar to those used to calculate lower bounds for ordinary Ramsey numbers (see, for instance, [13]). Fix n, and let us red/blue color the edges of K_n, giving any edge red or blue with probability 1/2, independent of the coloring of any other edges. For any size k subset S of $V(K_n)$, define the events

$$\begin{aligned} A_S &= \{E(S) \text{ has no blue triangle}\} \\ B_S &= \{E(S) \text{ has no red triangle}\} \\ \mathcal{B} &= \{K_n \text{ contains a monochromatic weight } k/2 \text{ half-clique}\} \\ &= \bigcup_{|S|=k} (A_S \cup B_S). \end{aligned}$$

Now, the probability that any given triangle is not all blue is 7/8, and any size k set S contains at least $(k-3)(k-4)/6$ edge-disjoint triangles by Lemma 3.8, so $\Pr\{A_S\} = \Pr\{B_S\} < (7/8)^{(k-3)(k-4)/6}$. So we have

$$\Pr\{\mathcal{B}\} = \Pr\left\{\bigcup_{|S|=k} (A_S \cup B_S)\right\} \leq 2 \sum_{|S|=k} \Pr\{A_S\} < 2 \binom{n}{k} \left(\frac{7}{8}\right)^{(k-3)(k-4)/6}.$$

If we choose n small enough so as to make this quantity less than 1, then $\Pr\{\mathcal{B}^c\} > 0$, so there must exist some edge 2-coloring of K_n with no monochromatic half-cliques of weight $k/2$. That is, there is some $K_n = H_1 \oplus H_2$ with $\bar{\omega}_2(H_i) < k/2$ for $i = 1, 2$. Since $\omega_2'(H_i) \leq 2\bar{\omega}_2(H_i)$, we know that $n \xrightarrow{2} (k, k)$ is false. □

Let us perform a rough asymptotic analysis of n versus k to find a lower bound on $r_2(k, k)$. If we take $\binom{n}{k} \approx n^k$ and $(7/8)^{(k-3)(k-4)/6} \approx (7/8)^{k^2/6}$, then

$$2\binom{n}{k}\left(\frac{7}{8}\right)^{\lfloor k(k-1)/6 \rfloor} < 1 \quad \text{becomes} \quad 2n^k\left(\frac{7}{8}\right)^{k^2/6} < 1.$$

Then $2n^k < (8/7)^{k^2/6}$, which gives, approximately, $n < (8/7)^{k/6}$. So we have the approximate bound of $r_2(k, k) > \left((8/7)^{1/6}\right)^k \doteq 1.0225^k$. While this is not nearly as good as the $\sqrt{2}^k$ lower bound for $r(k, k)$, we do know that $r(k, k) > r_2(k, k)$ in general, and this does at least establish the exponential growth of $r_2(k, k)$.

Since $\omega_b'(G) \to \omega_f(G)$ as $b \to \infty$ for any graph G (Theorem A.2), we wonder if the same holds for b- and fractional Ramsey numbers, i.e. does $r_b(x, y) \to r_f(x, y)$? Note that, since r_b and r_f are integer valued, convergence occurs iff there exists a $B \in \mathbf{N}$ for which $b \geq B$ implies $r_b = r_f$.

Theorem 3.10 *$r_b(x, y) \to r_f(x, y)$ as $b \to \infty$ if and only if (x, y) is not a discontinuity point of the function r_f.*

Proof. Recall $x, y, k, l, \varepsilon, \delta$ and $q = \min\{\lceil \varepsilon l \rceil, \lceil \delta k \rceil\}$ of Theorem 3.1. Notice that if we hold k and l fixed, ε and δ may range freely between multiples of $1/l$ and $1/k$, respectively, without changing the value of q. Thus $r_f(x, y)$ is constant over these rectangular regions of the plane. These rectangles are closed along their upper and right edges, and open on their lower and left edges. All discontinuity points of $r_f(x, y)$ lie on these edges, though not all points on these edges are discontinuity points. To be more precise, (x, y) is a discontinuity point of r_f if $r_f(x+\epsilon, y+\epsilon) > r_f(x, y)$ for all $\epsilon > 0$.

($\Longrightarrow$) **Lemma 3.11** *If (x, y) is a discontinuity point of r_f with $r_f(x, y) = n$, then there exists a decomposition $K_n = H_1 \oplus H_2$ such that $\omega_f(H_1) \leq x$ and $\omega_f(H_2) \leq y$.*

Proof. Suppose to the contrary. Then for every such decomposition, either $\omega_f(H_1) > x$ or $\omega_f(H_2) > y$. There are only a finite number of such decompositions, so let $\epsilon = \min\{\omega_f(H_1) - x, \omega_f(H_2) - y\}$, taken over all such positive values for all such decompositions. Thus for any decomposition $K_n = H_1 \oplus H_2$, we know that $\omega_f(H_1) \geq x + \epsilon$ or $\omega_f(H_2) \geq y + \epsilon$. But this tells us that $r_f(x+\epsilon, y+\epsilon) = n$, contradicting the fact that (x, y) is a discontinuity point of r_f. Our supposition to the contrary is therefore false. $\square$

For discontinuity point (x, y), take the decomposition $K_n = H_1 \oplus H_2$ indicated by Lemma 3.11. For $i = 1, 2$, let $\omega_f(H_i) = c_i/d_i$, where c_i/d_i is a lowest terms fraction. Let b be any positive integer such that neither d_1 nor d_2 divide b. Then there is no integer a_i such that $a_i/b_i = c_i/d_i$. Since $\omega'_b(G)$ must always be a multiple of $1/b$, we cannot have $\omega'_b(H_i) = c_i/d_i$. Thus

$$\omega'_b(H_1) < \omega_f(H_1) \leq x \quad \text{and} \quad \omega'_b(H_2) < \omega_f(H_2) \leq y$$

and so $r_b(x, y) > n = r_f(x, y)$. Since we may choose such b to be arbitrarily large, we have $r_b \not\to r_f$ at this (x, y).

($\Longleftarrow$) **Lemma 3.12** *For any graph G and $b \in \mathbf{N}$, $\omega'_b(G) > \omega_f(G) - |V(G)|/b$.*

Proof. Start with a maximum fractional clique, and round all weights on vertices down to the nearest multiple of $1/b$. The result is a valid b-clique, and total weight less than $|V(G)|/b$ has been removed. □

Let $r_f(x, y) = n$ be constant over all $x \in (x_1, x_2], y \in (y_1, y_2]$. Choose any specific $x \in (x_1, x_2)$ and $y \in (y_1, y_2)$, so that (x, y) is *not* a discontinuity point of r_f. Let $d = \min\{x_2 - x, y_2 - y\} > 0$, and choose $B \in \mathbf{N}$ such that $n/B < d$. Now, for any decomposition $K_n = H_1 \oplus H_2$, either $\omega_f(H_1) \geq x_2$ or $\omega_f(H_2) \geq y_2$. Then for any $b \geq B$, either

$$\omega_b'(H_1) > \omega_f(H_1) - n/b \geq x_2 - d \geq x_2 - (x_2 - x) = x$$

or

$$\omega_b'(H_2) > \omega_f(H_2) - n/b \geq y_2 - d \geq y_2 - (y_2 - y) = y,$$

so $n \xrightarrow{b} (x, y)$. Thus $r_b(x, y) = r_f(x, y)$ for all $b \geq B$, and so $r_b \to r_f$ at (x, y). □

Clearly, there is still much work that could be done in the area of b-Ramsey numbers. Even though $r_b(x, x)$ (presumably) grows exponentially in x, for almost any fixed x we have $r_b(x, x) \to r_f(x, x)$, a function which only grows quadratically in x. There seems to be some interesting ground to cover in comparing the growth rates of $r_b(x, y)$ in x and y versus b.

3.5 Lovász-ϑ Ramsey Numbers

The fractional clique number of a graph is a relaxation of the ordinary clique number; the Lovász-ϑ number of a graph, denoted $\vartheta(G)$, represents a weaker relaxation of clique number. That is,

$$\omega(G) \le \vartheta(G) \le \omega_f(G) \le \chi(G)$$

for any finite graph G. $\vartheta(G)$ was introduced by Lovász in 1977 (see [10]), and has been well studied since then (see [7] for an overview of history and results). We will define Lovász-ϑ Ramsey numbers by replacing clique number with the Lovász-ϑ number[17].

To define $\vartheta(G)$, we first need to define orthogonal labelings. An *orthogonal labeling* of a graph $G = (V, E)$ is an assignment of a unit[18] vector a_v to each $v \in V$ such that $a_u \cdot a_v = 0$ whenever $uv \in E$. That is, adjacent vertices are assigned perpendicular vectors, and $||a_v|| = 1$ for all $v \in V$. These vectors may be of any fixed dimension d. The *cost* of a vector in such a labeling is defined to be $c(a_v) = a_{1v}^2$, where a_{1v} is the first entry of a_v. The cost of the labeling, denoted $c(a)$, is just the vector of costs (whose v-th entry is $c(a_v)$).

Recall the integer **(ID)** and linear **(DP)** duals defining ω and ω_f from Section 1.1. Let us refer to the feasible regions of these programs as $\Omega(G)$ and $\Omega_f(G)$, respectively. We now define the region

$$\Theta(G) = \{y \in \mathbf{R}^n : c(a) \cdot y \le 1 \text{ for all orthogonal labelings } a \text{ of } G, \quad y \ge 0\}.$$

Similarly to ω and ω_f, we define

$$\vartheta(G) = \max \mathbf{1} \cdot \mathbf{y} \text{ s.t. } y \in \Theta(G).$$

It is easy to show[19] that $\Omega(G) \subseteq \Theta(G) \subseteq \Omega_f(G)$, from which $\omega(G) \le \vartheta(G) \le \omega_f(G)$

[17] Note: Our definitions of ϑ and orthogonal labelings of G are more commonly taken to be those of $\overline{G}$. We switch the roles of these two quantities to maintain consistency with our previous work. As Ramsey numbers are symmetric in the usage of G and $\overline{G}$, our choice of definitions will not affect our Ramsey results.

[18] Requiring *unit* vectors is not standard in the definition of orthogonal labelings, but is done without loss of generality for our purposes; see [7].

[19] See Lemma 2 from [7].

immediately follows.

We now take $n \xrightarrow{\vartheta} (x,y)$ to mean that, whenever $K_n = H_1 \oplus H_2$, we must have $\vartheta(H_1) \geq x$ or $\vartheta(H_2) \geq y$. Then the *ϑ-Ramsey number* $r_\vartheta(x,y)$ is the least positive integer n for which $n \xrightarrow{\vartheta} (x,y)$. While we cannot compute r_ϑ exactly as we did with r_f, we can show that it's value is nearly the same as r_f. Our result follows easily from the following lemma[20].

Lemma 3.13 *For a graph G on n vertices, $\vartheta(G)\vartheta(\overline{G}) \geq n$; if G is vertex transitive, then equality holds.* □

Theorem 3.14 $r_f(x,y) \leq r_\vartheta(x,y) \leq \lceil xy \rceil$.

Proof. Fix $x, y \geq 2$. Let $n = r_\vartheta(x,y)$, and let $K_n = H_1 \oplus H_2$ be any edge 2-coloring of K_n. Then $\omega_f(H_1) \geq \vartheta(H_1) \geq x$ or $\omega_f(H_2) \geq \vartheta(H_2) \geq y$, and so $n \xrightarrow{f} (x,y)$. Thus it follows that $r_f(x,y) \leq r_\vartheta(x,y)$.

Now take $n = \lceil xy \rceil$, and any decomposition $K_n = H_1 \oplus H_2$ (with $H_1 = \overline{H_2}$). Either $\vartheta(H_1) \geq x$ or $\vartheta(H_1) < x$ implies that $\vartheta(H_2) \geq \frac{n}{\vartheta(\overline{H_2})} > \frac{xy}{x} = y$ by Lemma 3.13. Thus $r_\vartheta(x,y) \leq \lceil xy \rceil$. □

Since $r_f(x,y)$ is very nearly as large as xy, we have fairly tight bounds on $r_\vartheta(x,y)$. It is interesting to note the closeness of the values of r_ϑ to r_f, even though ϑ lies somewhere between ω and ω_f.

Note that we did not use the second part of Lemma 3.13. If we could establish values of ϑ for a large class of vertex transitive graphs (as we did with $C_{n,m}$ for ω_f), we could likely achieve even more accurate bounds on r_ϑ.

[20] See Lemma 23 and Theorem 25 from [7].

4 Fractional Dimension of Posets from Trees

In this last chapter, we switch gears a little bit, and fractionalize the dimension of posets. We start with a few simple definitions to develop the language of posets. A binary relation on a set X is called a *partial order* if it is reflexive, antisymmetric, and transitive. A *partially ordered set* (or "poset") $P = (X, \leq)$ consists of some *ground set* X and a partial order $\leq$ on X. A pair of elements $x, y \in X$ are *incomparable* if $x \not\leq y$ and $y \not\leq x$. Such an incomparable (ordered) pair (x, y) is a *critical pair* if, for all $a, b \in X$ such that $a \leq x$ and $y \leq b$ (but not $(a, b) = (x, y)$), we have $a \leq b$. The idea of critical pairs plays an important role in this chapter, so we will let $C(P)$ (or just C) denote the set of critical pairs of P.

We may also think of $\leq$ as a subset of $X \times X$. A partial order L on X is a *total order* if for any $a, b \in X$, either $(a, b) \in L$ or $(b, a) \in L$. A total order L is a *linear extension* of $\leq$ if $\leq$ is a subset of L. A *realizer* of a poset P is a collection of linear extensions $\mathcal{R} = \{L_1, \ldots, L_t\}$ of $\leq$ whose intersection is $\leq$. That is, for any incomparable pair $x, y \in X$, there are $L_i, L_j \in \mathcal{R}$ with $(x, y) \in L_i$ and $(y, x) \in L_j$. Finally, the *dimension* of P is defined as the size of the smallest realizer of P, and is denoted $\dim(P)$.[21]

It is easily shown[22] that a collection of linear extensions $\{L_1, \ldots, L_t\}$ is a realizer if and only if, for every critical pair (x, y), there is some L_i containing (y, x). Even though x and y are incomparable, the critical pair (x, y) behaves as if $x \leq y$ relative to the rest of the poset, and so if $y < x$ in L_i, we say that L_i *reverses* (x, y). In other words, $\{L_1, \ldots, L_t\}$ is a realizer iff it reverses every critical pair.

[21] The concept of dimension of a poset was first introduced by Dushnik and Miller [3].
[22] See [14].

4.1 Fractional Dimension

We may now formulate dimension as an integer programming problem, just as we did chromatic number. For the poset $P = (X, \leq)$, let $\mathcal{L} = \{L_1, \ldots, L_m\}$ be the set of all linear extensions of $\leq$, and $C = \{c_1, \ldots, c_n\}$ the set of all critical pairs. Let M be the critical pair/linear extension incidence matrix, with rows indexed by C and columns indexed by $\mathcal{L}$. The i, j entry is a 1 exactly when c_i is reversed in L_j (henceforth denoted by $c_i \propto L_j$), and is 0 otherwise. Then, just as with ordinary/fractional chromatic number/clique number, we get pairs of integer and linear programs defining ordinary/fractional dimension and its dual parameter, which we denote $\kappa(P)$:

$$\begin{aligned}
\dim(P) &= \text{min } \mathbf{1}\cdot\mathbf{x} \quad \text{s.t. } M\mathbf{x} \geq \mathbf{1}, \quad \mathbf{x} \geq 0, \;\; \mathbf{x} \in \mathbf{Z}^m \\
\kappa(P) &= \text{max } \mathbf{1}\cdot\mathbf{y} \quad \text{s.t. } M^T\mathbf{y} \leq \mathbf{1}, \;\; \mathbf{y} \geq 0, \;\; \mathbf{y} \in \mathbf{Z}^n \\
\dim_f(P) &= \text{min } \mathbf{1}\cdot\mathbf{x} \quad \text{s.t. } M\mathbf{x} \geq \mathbf{1}, \quad \mathbf{x} \geq 0, \;\; \mathbf{x} \in \mathbf{R}^m \\
\kappa_f(P) &= \text{max } \mathbf{1}\cdot\mathbf{y} \quad \text{s.t. } M^T\mathbf{y} \leq \mathbf{1}, \;\; \mathbf{y} \geq 0, \;\; \mathbf{y} \in \mathbf{R}^n
\end{aligned}$$

Feasible solutions to the LPs are referred to as *fractional realizers* and *fractional critical pair packings* (or, henceforth, simply "fractional packings")[23], respectively. As before, $\kappa(P) \leq \kappa_f(P) = \dim_f(P) \leq \dim(P)$ for any finite poset P, and the middle "$=$" is only "$\leq$" if P is infinite.

Again, to facilitate our discussion, particularly with respect to infinite posets, we reformulate these definitions. Define a *fractional realizer* of P to be a mapping $f : \mathcal{L} \to [0, 1]$ such that for each $c \in C$ we have $\sum_{L \in \mathcal{L} : c \propto L} f(L) \geq 1$. The weight of this realizer is

[23] "Fractional critical pair packing" is, for lack of a better term, used in reference to the hypergraph covering/packing formulation presented in [11].

$w(f) = \sum_{L \in \mathcal{L}} f(L)$, and fractional dimension is

$$\dim_f(P) = \inf\{w(f) : f \text{ a fractional realizer of } P\}.$$

Again, this definition matches the linear programming formulation when P is finite, but is still well-defined when P is infinite.

We similarly modify the definition of fractional packing number: a *fractional packing* of P is a mapping $g : C \to [0, 1]$ such that for each $L \in \mathcal{L}$ we have $\sum_{c \propto L} g(c) \leq 1$. The weight of this mapping is $w(g) = \sum_{c \in C} g(c)$, and fractional packing number is

$$\kappa_f(P) = \sup\{w(g) : g \text{ a fractional packing of } G\}.$$

which is just the dual linear program formulation if P is finite.

We also have a b-fold version of fractional dimension, as follows: a collection of linear extensions $\{L_1, \ldots, L_t\}$ is a *b-fold realizer* if and only if every critical pair is reversed in (at least) b of the L_i's. Then the b-fold dimension of P, denoted $\dim_b(P)$, is the size of the smallest b-fold realizer of P. And, as before, we may also write $\dim_f(P) = \lim_{b \to \infty} \frac{\dim_b(P)}{b}$. The proof that the two definitions agree is analogous to that for χ_f, even in the case of infinite posets. A more complete treatment of fractional dimension may be found in Brightwell and Scheinerman [2].

4.2 Posets of Graphs

We may derive a poset $P(G)$ from a graph G as follows. The ground set is $X = V(G) \cup E(G)$, and the only (non-equality) relations are of the form $v < e$, where vertex v is an endpoint of edge e. What are the critical pairs of this poset?

For $u, v, w \in V(G)$, $vw \in E(G)$, if (u, vw) is a critical pair, then $a \leq u$ and $vw \leq b$ implies $a \leq b$. But $a \leq u$ implies $a = u$, and $vw \leq b$ implies $vw = b$, so $(a, b) = (u, vw)$. So if (u, vw) is an incomparable pair, there are no candidate a and b, and (u, vw) vacuously satisfies the definition of a critical pair. So any (vertex,edge) pair is a critical pair so long as the vertex isn't in the edge.

For $u, v \in V(G)$, if (u, v) is a critical pair, then $a \leq u$ and $v \leq b$ implies $a \leq b$. But $a \leq u$ implies $a = u$, so we require that $v \leq b$ implies $u \leq b$, that is, everything above v is also above u. Since all edges incident to v are above v, all these edges must also be above u. This can only happen if v is a leaf (vertex of degree 1) and u the other end of v's edge (u is henceforth referred to as a "branch"). This is the only possible type of (vertex,vertex) critical pair.

There are no (edge,vertex) critical pairs, for this would require each of the edge's vertices to be $\leq$ the critical pair vertex, which can't happen. Also, there are no (edge,edge) critical pairs, for this would require both of the first edge's vertices to be below the second edge, which only occurs if they are the same edge, and then not incomparable.

In order to establish the fractional dimension of one of our posets (or class of posets), we need to utilize the same approach we've seen before: bounding $\dim_f$ from above with fractional realizers, and bounding it below with fractional packings. Of course, for any given poset of a graph, there are a huge number of distinct linear extensions that must be considered in building a fractional realizer[24], so this is not a calculation we wish to undertake for most specific posets. We shall content ourselves with the limiting value of $\dim_f$ for certain classes of trees (including general trees). This is facilitated by the

[24] We shall see that there are, roughly, $|V(G)|!$ maximal linear extensions.

following.

Lemma 4.1 *Given graphs $G_1 \subset G_2$, it follows that*

$$\dim_f(P(G_1)) \leq \dim_f(P(G_2)).$$

Proof. Any incomparable pair in $P(G_1)$ is still an incomparable pair in $P(G_2)$ (adding vertices and edges can't add or alter the $v < uv$ relations between existing vertices and edges). So for an incomparable pair (a, b) in $P(G_1)$, this pair must have $b < a$ in linear extensions of total weight at least 1 in any fractional realizer of $P(G_2)$, and this fact holds when these linear extensions are restricted to elements of $P(G_1)$. Thus any fractional realizer of $P(G_2)$, when restricted to elements and relations is $P(G_1)$, is also a fractional realizer of $P(G_1)$. Given this, our desired result is immediate. □

Because of this, we limit our attention to posets of arbitrarily large graphs, and simply prove tight upper bounds. For instance, showing that $\lim_{q\to\infty} \dim_f(P(S_q)) = 1 + \sqrt{2}$ (where S_q is the q-star) proves that $1 + \sqrt{2}$ is a tight upper bound on $\dim_f$ for the posets of all finite stars.

A *maximal linear extension* is one for which the set of critical pairs it reverses is not a proper subset of the set of critical pairs reversed by any other linear extension. Just as we only needed to consider maximal independent sets for the fractional coloring problem (see Lemma 1.2), we need only consider maximal linear extensions for the fractional dimension problem:

Lemma 4.2 *The values of $\dim_f$ and κ_f don't change if we reformulate their definitions taking $\mathcal{L}$ to be the set of all* maximal *linear extensions.*

The proof is analogous to that of Lemma 1.2. Further, in our current environment, we have:

Lemma 4.3 *The critical pairs reversed in a maximal linear extension of the poset of any graph are fully determined by the ordering of $V(G)$ within it.*

Proof. We wish to show that, in constructing a maximal linear extension, once we've specified the order of the vertices, we can then specify where to place the edges without losing maximality. Given an edge $uv \in X$, the only two relations in P involving uv are $u < uv$ and $v < uv$, so uv must be placed somewhere above its endpoints in any linear extension. On the other hand, for the sake of reversing critical pairs, we want edges as low as possible, since edges are only in critical pairs of the form (vertex,edge), and this critical pair is only reversed if the edge is below the vertex in the linear extension. So the best place we can put uv is right above the higher of u and v in our linear extension. If several edges are placed directly above the same vertex, their relative order does not affect the reversal of any critical pairs. Thus, once we have ordered $V(G)$ in a linear extension, we know where we must place the edges if we wish to make this linear extension maximal. Further, the critical pair (u, vw) gets reversed exactly when u comes above both v and w in the ordering of $V(G)$. □

Henceforth, we will describe (maximal) linear extensions in terms of permutations of $V(G)$.

Schnyder [12] showed that $\dim(P(G)) \leq 3$ iff G is planar. For fractional dimension, on the other hand, Brightwell and Scheinerman [2] proved that $\dim_f(P(G)) \leq 3$ for *any* finite graph G, and that equality holds iff G contains a triangle. Taking trees to be the most obvious examples of triangle-free graphs, they went on to show that $\dim_f(P(T)) \leq 1+\phi \doteq 2.61803$ for any tree T, where $\phi = \frac{1}{2}(1+\sqrt{5})$ is the golden mean, and conjectured that this upper bound was tight. In the remainder of this chapter, we will show that the correct value of this upper bound is (approximately) 2.44504, and present other specific results pertaining to stars, binary trees, and infinite trees.

4.3 Posets of Trees and $\dim_f$ for Posets of Stars

We shall consider only complete, rooted q-ary trees, that is, trees where every non-leaf vertex has q children, every non-root vertex has one parent, and all leaves are at the same level of the tree. Any tree may be considered rooted, and if T has maximum vertex degree Δ, then it is a subtree of some complete, rooted $(\Delta - 1)$-ary tree. By Lemma 4.1, this is sufficient to establish an upper bound on $\dim_f(P(T))$. Henceforth, n will denote the depth of any tree under consideration.

We warm up with the relatively simple calculation of a tight upper bound for $\dim_f$ of the posets of finite stars. This result was noted, but not proved, by Brightwell and Scheinerman [2].

Theorem 4.4 $\lim_{q\to\infty} \dim_f(P(S_q)) = 1+\sqrt{2}$

Proof. Since we can describe our linear extensions of $P(S_q)$ with permutations of $V(S_q)$, there is, up to isomorphism, only one decisions to make in constructing them: how far down to put the root. Let us put a fraction $p = 2-\sqrt{2}$ of the leaves above the root, and let L_p be the random variable which chooses a linear extension uniformly at random from the set of all such linear extensions. In the star with root r, there are only two types of critical pairs: (r, u) and (u, rv).

$$\begin{aligned} \Pr\{L_p \text{ reverses } (r,u)\} &= \Pr\{r > u \text{ in } L_p\} = 1 - p = \sqrt{2} - 1, \\ \Pr\{L_p \text{ reverses } (u,rv)\} &= \Pr\{u > r, v \text{ in } L_p\} \\ &\approx p((1-p) + p/2) = p - p^2/2 = \sqrt{2} - 1. \end{aligned}$$

Note that the quantity $p((1-p)+p/2)$ assumes that each leaf is being put over the root independently with probability p. Although this is never the case with finite stars, this approximation becomes arbitrarily close to correct as $n \to \infty$, and so will serve in our limit calculations.

Now, if we distribute total weight $1/p$ evenly among all linear extensions in the sample space of L_p, then the total weight on any critical pair is

(total weight on L_p)·(fraction of L_p containing the critical pair) $= (1/p) \cdot (p) = 1$.

Thus this weighting creates a valid fractional realizer of weight $1/p = 1 + \sqrt{2}$. Of course, we can never actually take an exact portion $2 - \sqrt{2}$ of leaves, but we may get arbitrarily close to this as q gets large, so we have our desired upper bound on the limit.

For a lower bound, let L_p be as above, but let p take on any value in [0,1]. Now, the probablitity that a given critical pair will be reversed by L_p is the same as the fraction of that type of critical pair that are reversed by any one member of L_p's sample space. So if we distribute total weight α evenly among all (r, v) critical pairs, and β among all (u, rv) critical pairs, then the total weight put on any linear extension from L_p is

$$w_{\alpha,\beta}(p) = \alpha(1-p) + \beta(p - p^2/2) \ , \quad \text{and}$$

$$\frac{d}{dp}w_{\alpha,\beta}(p) = \beta(1-p) - \alpha = 0 \quad \text{at} \quad p = 1 - \frac{\alpha}{\beta}$$
$$\frac{d^2}{dp^2}w_{\alpha,\beta}(p) = -\beta$$

So $w_{\alpha,\beta}(p)$ attains its maximum value at $p = 1 - \alpha/\beta$. As in our upper bound calculations, the quantity $(p - p^2/2)$ is only actually correct if we are placing each leaf above the root independently with probability p, but again, it becomes arbitrarily close to correct in the limit, which is all we are presently concerned with.

If we set $\alpha = \sqrt{2}/2$ and $\beta = (\sqrt{2}+1)/\sqrt{2}$, then w attains its maximum value for $p = 2 - \sqrt{2}$, and

$$w_{\alpha,\beta}(2-\sqrt{2}) = \frac{\sqrt{2}}{2}(\sqrt{2}-1) + \frac{\sqrt{2}+1}{\sqrt{2}}(\sqrt{2}-1) = 1$$

So our weighting of critical pairs never puts weight more than 1 on any linear extension, and so is a valid fractional packing. The value of this fractional packing is $\alpha+\beta = 1+\sqrt{2}$, giving the desired lower bound. Note that it is the inaccuracy in the $(p-p^2/2)$ value which keeps this from actually being a valid fractional packing in any finite case. It is, however, sufficient in showing that, as $q \to \infty$, we can create fractional packings arbitrarily close to this value, which is all we need to establish our limit. □

Note that, since $S_{q-1} \subset S_q$, Lemma 4.1 tells us that $\dim_f(P(S_q))$ is an increasing function of q, and so actually *increases* to the limiting value of $1+\sqrt{2}$.

We now move on to trees of arbitrary depth. Henceforth, for the sake of our optimal fractional realizers and packings, it suffices to consider very specific classes of linear extensions and critical pairs. A linear extension of a poset of a tree is *contiguous* if, for any vertex x with children $y_1, \ldots, y_q$, all the vertices of any subtree rooted at some y_i appear consecutively in the linear extension. That is, if T_i is the subtree rooted at y_i, then a contiguous linear extension has

$$\ldots, V(T_1), V(T_2), \ldots, V(T_k), x, V(T_{k+1}), \ldots, V(T_q), \ldots$$

appearing consecutively in it, for some ordering of x's children and some $k \in \{0, 1, \ldots, q\}$. Any vertex not in the subtree rooted at x comes before or after these vertices in the linear extension, and the vertices of each $V(T_i)$ appear within this collection in a similarly contiguous fashion. See Figure 7(b).

A more constructive description comes from building the tree recursively. If we build the tree by recursively replacing each leaf by a q-star with that leaf as the center, then we similarly update the linear extension by replacing the leaf with the entire star contiguous within the linear extension.

Since we are talking about complete q-ary trees, we may now fully describe contiguous linear extensions, up to isomorphism, by specifying what fraction of each vertex's children appear above it in the linear extension. We shall be more precise about this below.

In the case of critical pairs, we don't limit our consideration to a select few, as we did with linear extensions, but instead observe that every critical pair can be put into one of just a few categories. In any contiguous linear extension, this category fully describes a critical pair's behavior. Of course, all (branch,leaf) critical pairs are identical up to isomorphism. The (vertex,edge) critical pairs may be described by where the vertex falls in the rooted tree relative to the edge; specifically, what is the lowest common ancestor of the vertex and the lower endpoint of the edge. For the critical pair (u, vw), where v is above w in the tree,

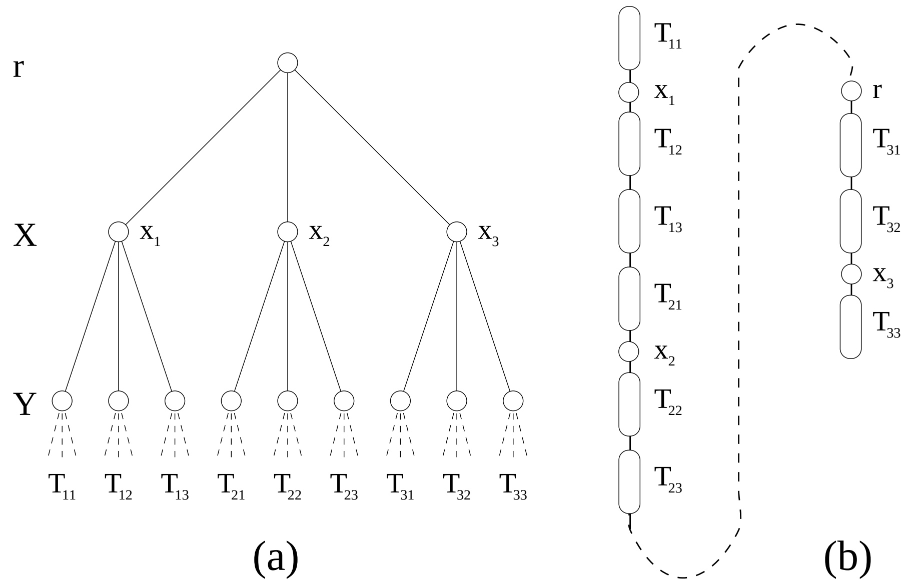

Figure 7: (a) The top three levels of a complete ternary tree. (b) A contiguous linear extension thereof, where each T_{ij} represent the entire subtree rooted at a level Y vertex. Note that the subtree rooted at each x_i appears contiguously within this linear extension.

we have four choices for the lowest common ancestor of u and w:

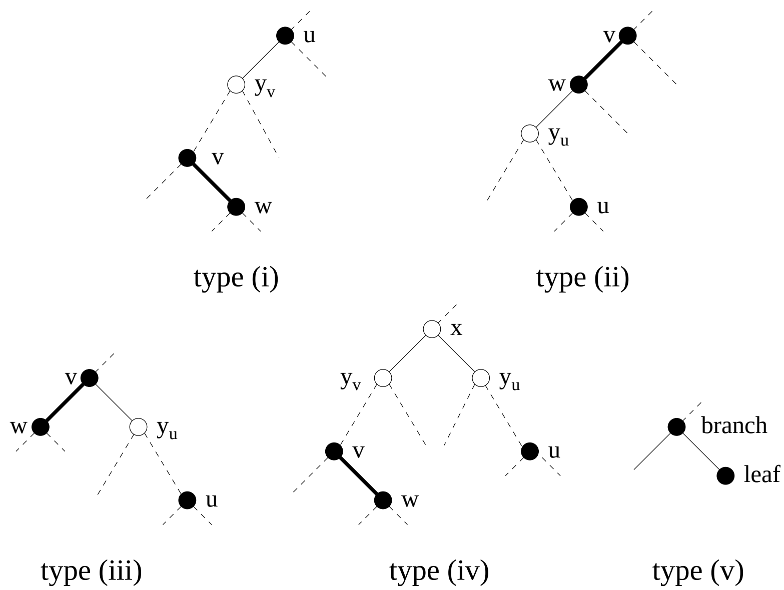

Figure 8: The five types of critical pairs and their intermediate vertices

(i) u: vw is in the subtree rooted at u (vw is below u in T)

(ii) w: u is in the subtree rooted at w (u is below v and w in T)

(iii) v: u is below v and one of w's "siblings" (not beneath w)

(iv) none of the above: u and vw are in different branches of some subtree

(v) we shall henceforth use this designation for (branch,leaf) critical pairs.

We may now fully characterize the conditions necessary for each such critical pair type to be reversed in a contiguous linear extension (putting vertex>edge or branch>leaf; see Figure 8):

(i) if y_v is the child of u whose subtree contains vw (possibly $y_v = v$), we must have $u > y_v$ (the entire subtree rooted at y_v, including vw, is then beneath u)

(ii) if y_u is the child of w whose subtree contains u (possibly $y_u = u$), we must have $w > v$ (to get w's subtree, including u, above v) and $y_u > w$ (to get y_u's subtree, including u, above w)

(iii) if y_u is the child of v whose subtree contains u (possibly $y_u = u$), we need $y_u > v$ and either $v > w$ or $y_u > w > v$.

(iv) if x is u and v's lowest common ancestor, and y_u and y_v are x's children whose subtrees contain u and vw, respectively (possibly $y_u = u$ or $y_v = v$), we must have $y_u > y_v$ (x's relative position is irrelevant).

(v) we must simply have branch>leaf.

If we chose a contiguous linear extension "at random", then the probability of any of the above events can be described in terms of the fraction of vertices' children which appear above them.

We now present two results. Because the proofs of each are similar, we state the results first, then dedicate separate sections to all upper bound and tighness calculations.

Theorem 4.5 *For any (finite or infinite) binary tree T (i.e. tree with maximum degree 3), $\dim_f(P(T)) \leq \frac{7}{3}$, and this bound is best possible.*

Theorem 4.6 *For any (finite or infinite) tree T of bounded degree, $\dim_f(P(T)) \leq z_0 \approx 2.44504$, where z_0 is a root of $z^3 - 7z^2 + 14z - 7 = 0$. This bound is, at least to within 2000 decimal places of accuracy, best possible.*

4.4 Upper Bound Calculations

As mentioned earlier, we may fully describe contiguous linear extensions simply by stating what fraction of a vertex's children are above it in the linear extension. The simplest way to do this is to let this fraction be the same for all vertices. In particular, for a q-ary tree, we may define a contiguous linear extension L_i, up to isomorphism, by allowing every vertex to have exactly i of its children above it in L_i, for any $i \in \{0, \ldots, q\}$. We may also treat L_i as a random variable: we choose a linear extension uniformly at random from the set of all such L_i's. Note that the arrangement of a vertex's children around it is independent of that for any other vertex. Then the probability that any vertex is above its parent in L_i is $\frac{i}{q}$. Further, we can calculate the exact probability that any of our five types of critical pairs is reversed in L_i. Again, for the critical pair (u, vw), we have:

(i) Pr$\{u > y_v\} = \frac{q-i}{q}$

(ii) Pr$\{y_u > w > v\} = \frac{i^2}{q^2}$

(iii) Pr$\{y_u > v$ and $y_u > w\} = \frac{\sum_{k=1}^{i}(q-k)}{q(q-1)} = \frac{i(2q-i-1)}{2q(q-1)}$

(iv) Pr$\{y_u > y_v\} = \frac{1}{2}$

(v) Pr$\{$branch>leaf$\} = \frac{q-i}{q}$

The optimal usage of such L_i's in constructing fractional realizers requires a mix of different i values. If we use L_i a fraction a_i of the time, where $\sum_i a_i = 1$, then the above becomes

(i) Pr$\{u > y_v\} = \sum_{i=0}^{q} \left(\frac{q-i}{q}\right) a_i$

(ii) Pr$\{y_u > w > v\} = \sum_{i=0}^{q} \left(\frac{i^2}{q^2}\right) a_i$

(iii) $\Pr\{y_u > v \text{ and } y_u > w\} = \sum_{i=0}^{q} \left(\frac{i(2q-i-1)}{2q(q-1)}\right) a_i$

(iv) $\Pr\{y_u > y_v\} = \frac{1}{2}$

(v) $\Pr\{\text{branch}>\text{leaf}\} = \sum_{i=0}^{q} \left(\frac{q-i}{q}\right) a_i$

Suppose that the smallest value above is p; that is, every critical pair gets probability weight at least p. Then if we actually put weight a_i/p on L_i [25] for each i, the total weight used is $1/p$. Further, each of the values in (i)-(v) above gets multiplied by $1/p$, and so is at least 1. We then have a fractional realizer of weight $1/p$, and so we wish to maximize p to get the best upper bound possible. Note that (i) and (v) are identical, and since we will not be able to get all of (i)(ii)(iii) simultaneously above $\frac{1}{2}$, (iv) will never be our smallest value. Trying to get the minimum of (i)-(iii) as large as possible is a balancing act, where making one larger makes another smaller. So the best we can do is when they're all equal[26]. So we may write each of (i)-(iii) above as $\sum_i c_i a_i = p$ for the appropriate values of the c_i's. Along with $\sum_i a_i = 1$, we have a linear system with 4 equations and $q+1$ unknowns. Therefore, so long as the system is consistent, a solution will exist with only 4 non-zero variables. We clearly need p to be one of these; let the others be a_i, a_j, and a_k. Since $a_k = 1 - a_i - a_j$, we can make this substitution in (i)-(iii) and remove the last equation. Rearranging, (i)-(iii) take the form

$$(c_i - c_k)a_i + (c_j - c_k)a_j - p = -c_k \ .$$

Putting this in matrix form, substituting in the appropriate c values for (i)-(iii) and

[25] More specifically, if we distribute total weight a_i/p evenly among all linear extensions from L_i's sample space

[26] This claim is made without proof; such a proof adds no real content, as what follows would be valid even if it were false. We still establish a valid upper bound.

collecting these in a single matrix equation, we get

$$\begin{bmatrix} \frac{k-i}{q} & \frac{k-j}{q} & -1 \\ \frac{i^2-k^2}{q^2} & \frac{j^2-k^2}{q^2} & -1 \\ \frac{2q(i-k)+i(i+1)+k(k+1)}{2q(q-1)} & \frac{2q(j-k)+j(j+1)+k(k+1)}{2q(q-1)} & -1 \end{bmatrix} \begin{bmatrix} a_i \\ a_j \\ p \end{bmatrix} = \begin{bmatrix} -\frac{q-k}{q} \\ -\frac{k^2}{q^2} \\ -\frac{k(2q-k-1)}{2q(q-1)} \end{bmatrix}$$

If we represent the above as $Ax = b$, we may find x explicitly as $A^{-1}b$, since $A^{-1} = \frac{1}{\det A}\text{adj}(A)$ may be calculated symbolically. Interestingly, regardless of our choice of (i, j, k), p always solves as $\frac{2q-1}{5q-3}$, giving a fractional realizer weight of $\frac{5q-3}{2q-1}$. We need only check that we can find some (i, j, k) where $a_i, a_j, a_k \geq 0$.[27] It turns out that a number of choices of (i, j, k) work. In particular, $(i, j, k) = (0, \lceil q/2 \rceil, q)$ always works. In the case of q even, for example, we get

$$(a_i, a_j, a_k) = \left(\frac{1}{5q-3}, \frac{4q-4}{5q-3}, \frac{q}{5q-3} \right)$$

Multiplying these values by $1/p = \frac{5q-3}{2q-1}$ gives the actual amount of weight we wish to assign to each type of linear extension. This assignment constitutes a valid fractional realizer of weight $\frac{5q-3}{2q-1}$, which proves $\dim_f(P(T)) \leq \frac{5q-3}{2q-1}$ for any complete, finite q-ary tree T, and thus for *any* finite tree with maximum degree $q+1$.

In a finite tree, there are only a finite number of linear extensions with parameter i, and so it makes sense to take the weight a_i/p and divide it evenly among all linear extensions in the class L_i. In the case of infinite trees, however, this clearly does not work. We may, however, through a clever choice of a finite collection of members of L_i, create the desired

[27] So that all linear extensions receive non-negative weights, and we actually do have a valid fractional realizer.

randomness and independence in putting vertices above or below their parents, even in an infinite tree. Let Π be the set of permutations on $[q]$. For $\pi, \sigma \in \Pi$, define the contiguous linear extension $L_i(\pi, \sigma)$ as follows. If vertex v is on an odd level of T, arrange its children in $L_i(\pi, \sigma)$ according to π; if v is on an even level, arrange its children according to σ; in either case, put v below exactly i of its children. We now chose the random variable L_i uniformly from $\{L_i(\pi, \sigma) : (\pi, \sigma) \in \Pi \times \Pi\}$. Now not only are all possible arrangements of a vertex's children equally likely, but they are independent of the arrangement of that vertex's siblings. Note that the reversal of any critical pair is determined by the ordering of vertices from (at most) three consecutive levels of T. For example, whether or not a type(ii) critical pair (u, vw) is reversed is fully determined by the location of v, w and y_u within $L_i(\pi, \sigma)$, and all these vertices lie within three consecutive levels of T. Since the arrangement of the higher of these levels is independent of the arrangements within the lower level, all our previous calculations are valid for this construction. Thus our result also applies to infinite q-ary trees.

We notice that, as $q \to \infty$, $1/p = \frac{5q-3}{2q-1}$ approaches 2.5, which is not the best possible upper bound for posets of trees. However, this does provide the best *known* upper bound for instances of q where this value is less than then general bound of 2.44504 (specifically, $q = 2, 3, 4, 5$).[28] In the case of binary trees $(q = 2)$, this bound is best possible.

Proof of Theorem 4.5 (first part). Setting $q = 2$ in the above informs us that, for all (finite or infinite) binary trees T, $\dim_f(P(T)) \leq \frac{7}{3}$. $\square$

The proof that this bound is best possible appears in the following section.

Since these bounds are not, in general, best possible, how may we improve on them?

[28]Note that $q = 5$ gives $1/p = 2.\overline{4}$ but $q = 6$ gives $1/p = 2.\overline{45}$.

So far, we have taken the probability that a vertex appears below one of its children to be the same for all vertices. If instead we make this probability conditional on whether or not a vertex is itself above or below its parent, we may improve our limiting bound.

Proof of Theorem 4.6 (first part). Given the poset of a complete q-ary tree, we wish to define a single class of contiguous linear extensions, all of which are equivalent. We let L be a random variable which chooses one such linear extension uniformly at random. We now build our contiguous linear extensions "top down", i.e. recursively, starting at the top. For any vertex y with parent x and child z, let

$$a = \text{Pr}\{z > y \text{ in } L\}$$
$$b = \text{Pr}\{z > y \text{ in } L \mid y > x \text{ in } L\}$$
$$c = \text{Pr}\{z > y \text{ in } L \mid y < x \text{ in } L\}$$

In order to make sure that these three are consistent, we will require that

$$\text{Pr}\{z > y\} = \text{Pr}\{z > y|y > x\}\text{Pr}\{y > x\} + \text{Pr}\{z > y|y < x\}\text{Pr}\{y < x\}$$

i.e. that $a = ab + (1 - a)c$. Also, while every other vertex will have either a fraction b or c of its children above it, the root will actually have a fraction a of its children above it. Now, as before, we may calculate the probability that each type of critical pair gets reversed in L. For the critical pair (u, vw), we have:

(i) $\text{Pr}\{u > y_v\} = 1 - a$, or equivalently, $a(1 - b) + (1 - a)(1 - c)$

(ii) $\text{Pr}\{y_u > w > v\} = ab$

(iii) $\Pr\{y_u > v \text{ and } y_u > w\} = a(b-b^2/2)+(1-a)(c-c^2/2)$, or just $a-a^2/2$ if v is the root

(iv) $\Pr\{y_u > y_v\} = \frac{1}{2}$

(v) $\Pr\{\text{branch}>\text{leaf}\} =$ same as (i)

As before, we wish to maximize the minimum of these quantities (which we call p), and we do so by setting (i)=(ii)=(iii). Using both quantities in (i), we get three equations, which solve as

$b = \frac{1-a}{a}$

$c = \frac{2a-1}{1-a}$

$0 = 7a^3 - 7a^2 + 1$

the last of which has two roots in $[0,1]$, but the larger gives $c \doteq 1.8019$, which is not in $[0,1]$. We are left with the following solution:

$$a \doteq .59101, \quad b \doteq .69202, \quad c \doteq .44504, \quad p \doteq .40899, \quad 1/p \doteq 2.44504.$$

And, as desired, this solution satisfies our requirement that $a = ab + (1-a)c$. As before, we distribute total weight $1/p$ evenly over the sample space of L, and since every critical pair is in a fraction at least p of these, each critical pair receives total weight at least 1. We thus have a valid fractional realizer of weight $1/p$. This is exactly the value we want for z_0. Noting that $z_0 = 1/p = 1/(1-a)$, we have $a = (z_0-1)/z_0$, and substituting this into $0 = 7a^3 - 7a^2 + 1$ gives $z_0^3 - 7z_0^2 + 14z_0 - 7 = 0$. So $\dim_f(P(T)) \leq z_0$ for any tree in question.

Of course, a, b and c are irrational, so we can never put these exact fractions of children over their parents. However, as q gets large, we can get arbitrarily close to these values. Further, since $T_1 \subset T_2$ implies $\dim_f(P(T_1)) \leq \dim_f(P(T_2))$, $\dim_f$ has to be a non-decreasing function of q, and so this upper bound applies to *all* complete rooted trees. We can use the same technique previously described to reduce the size of L's sample space to $(q!)^2$ for infinite trees, so long as q is finite. Thus we have our upper bound on all (finite or infinite) complete q-ary trees for any finite q, and thus the bound also applies to all trees of bounded degree. □

Again, the proof that this bound is best possible appears in the following section.

4.5 The Tightness of Upper Bounds

We now establish that the given upper bounds are best possible. To this end, we consider the dual invariant, fractional packing number, since the value of *any* valid fractional packing serves as a lower bound on fractional dimension. We proceed by assigning weights to all critical pairs, and then establishing that we have a fractional packing by means of a dynamic program. We are still working with complete q-ary trees (q fixed), and we wish to construct the depth n tree T_n recursively as follows: T_2 is just the q-star (with its center the root); to make T_{n+1}, we take q copies of T_n, create a new root, and draw edges between the new root and each of the old roots. We only assign weight to "close-as-possible" critical pairs; that is, for the (vertex,edge) critical pairs of any particular type, we require the vertex to be as close as possible to the edge. In terms of our previous language, we only use critical pairs where $y_u = u$ or $y_v = v$. So type (i) and (ii) critical pairs are contained within three consecutive levels of the tree, and type (iii) are within two. "Close-as-possible" is redundant for type (v), and we never assign weight to type (iv). At each iteration, the "new" critical pairs are exactly the ones that use the new root (either as a vertex or edge endpoint). All other critical pairs are (copies of) ones found in previous iterations. For convenience, we refer to the top three levels of the tree as r (the root), X and Y, in downwards order (see Figure 7(a)), and thus we only directly discuss critical pairs involving vertices in these levels. For each type of critical pair except (iv), we associate a fixed weight:

(i): γ (ii): δ (iii): β (v): α

This weight is distributed evenly among all new critical pairs of the indicated type. So, for example, since there are $q(q-1)$ new type (iii) critical pairs at each iteration, we assign to each such new critical pair a weight of $\beta/(q(q-1))$. When we iterate, all old weights are divided by q, since we have made q copies of all critical pairs. This way, while the weight on any particular (copy of an) old critical pair is divided by q, the total weight on all old critical pairs remains unchanged. Specifically, if we define

$w_n =$ total weight on all critical pairs in the tree T_n,

we may recursively calculate[29] $w_n = w_{n-1} + \beta + \delta + \gamma$. Of course, at any step, we would have to scale down all weights so that no linear extension received total weight more than 1, but for now we address this issue only indirectly. At each stage we wish to know, "What is the most weight that this critical pair weighting puts on any linear extension of $P(T_n)$?" We condition this answer on the number of sub-roots (children of the root) that appear above the root in the linear extension in question, and therein lies our dynamic program. We define

[29] Because they involve leaves, there are never any new type (v) critical pairs after T_2, so we do not add α.

$f_n(i) =$ maximum possible weight on any linear extension of $P(T_n)$ that has i subroots above the main root.

Looking at the different types of critical pairs (excepting type (iv), which always get weight 0), we see that only type (iii) uses vertices from more than one T_{n-1} subtree. So within a linear extension, once we have ordered the root and its children, we only care about the ordering of the other vertices relative to other vertices in their own T_{n-1} subtree; how vertices from different subtrees are mixed is immaterial. Further, for any new type (i) or (ii) critical pairs, we need only consider a single T_{n-1} subtree, and whether *its* subroot is placed above or below the main root. We define

$g_n^x =$ maximum possible total weight that can be put on critical pairs in any $(T_{n-1} + \text{root})$ by a linear extension where the subtree's root is above the main root.

$g_n^r =$ maximum possible total weight that can be put on critical pairs in any $(T_{n-1} + \text{root})$ by a linear extension where the subtree's root is below the main root.

These values are, roughly, $f_{n-1}(i)/q$ plus weight added by new type (i) and (ii) critical pairs. Specifically, for each such subtree, we wish to use the i which gives the largest possible weight. For a linear extension of (a copy of) T_{n-1}, if i of *it's* subroots (in Y) are above its main root (in X), then we can put at most $f_{n-1}(i)/q$ total weight on it. If its root is above the root r of T_n, then no new type (i) critical pairs may be reversed (reversing these requires r to be placed *above* the subtree's root). Further, a type (ii) critical pair is reversed only if its Y vertex is above both its X parent and r. We have specified that i such Y vertices are above their parent (in this copy of T_{n-1}), which is in turn above r, so exactly i type (ii) critical pairs (from a total of q) are reversed by this linear extension of the $T_{n-1}+r$ subtree. Since each of the q^2 new type (ii) critical pairs gets weight δ/q^2, we have

$$g_n^x = \max_{i=0,\ldots,q}\left\{\frac{1}{q}f_{n-1}(i) + \frac{i}{q^2}\delta\right\}$$

Applying a similar analysis to the case where the root of T_{n-1} (in X) is below r, we see that $(q-i)$ of the Y vertices go below their parent, so for each of these, the corresponding type (i) critical pair is reversed. For the i vertices in Y that go above their parent in X, we may freely put them between r and their parent, or above r. The former reverses a type (i) critical pair, while the latter reverses a type (ii). Since these are the only critical pairs in $T_{n-1}+r$ that are affected by the positioning of r relative to $V(T_{n-1})$, we make this decision based on which of δ or γ is larger. Then we have

$$g_n^r = \max_{i=0,\ldots,q}\left\{\frac{1}{q}f_{n-1}(i) + \frac{q-i}{q^2}\gamma + \frac{i}{q^2}\cdot\max\{\delta,\gamma\}\right\}$$

We may now compute f_n by considering which type (iii) critical pairs are reversed. For each X vertex u above the root, every type (iii) critical pair of the form (u, rw) in which u is above w is reversed. There are $\sum_{k=1}^{i}(q-k) = (iq - i(i+1)/2)$ of these reversed (from a total of $q(q-1)$ type (iii)'s), and so

$$f_n(i) = \left(\frac{iq - i(i+1)/2}{q(q-1)}\right)\beta + i\cdot g_n^x + (q-i)\cdot g_n^r$$

Finally, we have

$$\begin{aligned}
f_2(i) &= \left(\frac{iq - i(i+1)/2}{q(q-1)}\right)\beta + \frac{q-i}{q}\alpha \\
w_2 &= \beta + \alpha \\
w_n &= w_{n-1} + \beta + \delta + \gamma
\end{aligned}$$

and so we have established the mechanics of our dynamic program. For convenience, define $f_n = \max_{i=0,\dots,q} f_n(i)$. Now, after the nth iteration, when we have used the indicated weighting scheme and determined that no linear extension of $P(T_n)$ can have total weight more than f_n, if we divide all the weights of all the critical pairs by f_n, then we have a valid fractional packing of $P(T_n)$ with value w_n/f_n. This value is a lower bound on $\dim_f(P(T_n))$. Further, it is a lower bound for *any* non-negative values of α, β, δ and γ, so the best lower bound comes with the best selection of these values. With this in mind, we are ready to establish a lower bound for $\dim_f$ of all posets of binary trees.

Proof of Theorem 4.5 (second half). We have $q = 2$, and all notation as above. Set

$$\alpha = 2, \quad \beta = 2, \quad \delta = 2, \quad \gamma = 3$$

Since $\gamma > \delta$, we have

$$f_2(0) = \alpha = 2, \quad f_2(1) = \beta/2 + \alpha/2 = 2, \quad f_2(2) = \beta/2 = 1$$

$$g_n^x = \max_{i=0,1,2} \left\{ \frac{1}{2} f_{n-1}(i) + \frac{i}{2} \right\}$$

$$g_n^r = \max_{i=0,1,2} \left\{ \frac{1}{2} f_{n-1}(i) + \frac{3}{2} \right\}$$

$$f_n(0) = 2g_n^r, \quad f_n(1) = 1 + g_n^x + g_n^r, \quad f_n(2) = 1 + 2g_n^x$$

$$w_2 = 4, \quad w_n = w_{n-1} + 7$$

Notice that, if $f_{n-1}(0) = f_{n-1}(1) = f_{n-1}(2) + 1$, then $f_n(1) - f_n(2) = g_n^r - g_n^x = 1$ and $f_n(0) - f_n(1) = g_n^r - g_n^x - 1 = 0$. Since these relations hold for the $n = 2$ base case, they must hold for all n inductively. So $f_n = f_n(0) = 2g_n^r = f_{n-1}(0) + 3 = f_{n-1} + 3$ for all $n \geq 3$. So we have

$$w_n = w_2 + 7(n-2) = 7n - 10 \text{ and } f_n = f_2 + 3(n-2) = 3n - 4$$

Since w_n/f_n serves as a lower bound on $\dim_f$, we have that $\dim_f(P(T_n)) \geq \frac{7n-10}{3n-4}$ for a complete n-level binary tree T_n, and so the previously derived upper bound of $7/3$ is best possible. □

Proof of Theorem 4.6 (2nd half). As with the upper bound, in order to establish a lower bound for *all* trees, we want $q \to \infty$. Notice that, although the argument i of $f_n(i)$ is a *number* of vertices, in the formulas it generally appears as part of a *fraction* of critical pairs. So let us reformulate $f_n(i)$ slightly, so that now i represents the *fraction* of the root's children which appear above it in a linear extension. Now, the domain of $f_n(i)$ is $\{0, \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q}{q}\}$, so as $q \to \infty$, we may choose arguments arbitrarily close to any real value in $[0, 1]$. Thus we may approximate this situation simply by taking $f_n(i)$ to be a function on the real interval $[0, 1]$. We next reformulate our entire dynamic program in this continuous form, which will demonstrate the limiting behavior of our previous system as $q \to \infty$.

We assume that we may choose *any* real fraction i of a vertex's children to go above it in a linear extension. We still talk about T_n as if q were finite, for if we actually had $q = \infty$, it would make no sense to talk about a specific fraction of an infinite number of children. We still iterate our tree and functions in the same manner, and still distribute weights α, β, δ and γ evenly on the various types of "new" critical pairs. $f_n(i)$ is still the most weight put on any linear extension of $P(T_n)$ with parameter i by our weighting. However, we must slightly reformulate the meanings of g_n^x and g_n^r:

$g_n^x =$ maximum possible total weight that can be put on all critical pairs in all $(T_{n-1} +$

root)'s by a linear extension where every subtree's root is above the main root.

$g_n^r =$ maximum possible total weight that can be put on all critical pairs in all $(T_{n-1} +$ root$)$'s by a linear extension where every subtree's root is below the main root.

These values still represent weight that lies only within T_{n-1}+root subtrees, but now each represents the weight on *all* such subtrees, were the root to fall as indicated for each of them. But since the weight on any collection of such subtrees is independent of the weight on any other disjoint collection, if exactly a fraction i of r's children come above it, then exactly a fraction i of the total weight represented by g_n^x will be on this linear extension (as well as $1 - i$ of the total weight of g_n^r). We may now formulate our expressions much the same as before, though it is much easier to represent the fraction of a particular type of critical pair which is being reversed[30]:

$$\begin{aligned}
g_n^x &= \max_{i \in [0,1]} \{f_{n-1}(i) + i\delta\} \\
g_n^r &= \max_{i \in [0,1]} \{f_{n-1}(i) + (1-i)\gamma + i \cdot \max\{\delta, \gamma\}\} \\
f_2(i) &= (i - i^2/2)\beta + (1-i)\alpha \\
f_n(i) &= (i - i^2/2)\beta + i g_n^x + (1-i) g_n^r \\
w_2 &= \beta + \alpha \\
w_n &= w_{n-1} + \beta + \delta + \gamma
\end{aligned}$$

We henceforth take $\gamma \geq \delta$, so that $g_n^r = \gamma + \max_i\{f_{n-1}(i)\}$. By choosing α carefully, we can force $f_n(i)$ to behave nicely.

[30] The $i - i^2/2$ of the f formulations is exactly the same as the expression $p - p^2/2$ found in the proof for stars. It is the fraction of new type (iii) critical pairs that are reversed when a fraction i of the subroots are placed above the main root.

Lemma 4.7 *If we choose* $\alpha = \frac{\beta\gamma - \beta\delta + \delta^2/2}{\beta - \delta}$*, then* $f_n(i) = f_{n-1}(i) + c$ *for all* $i \in [0, 1]$*, where* $c = g_3^r - \alpha$*.*

Proof. Let j =argmax(g_3^x) and k =argmax(g_3^r). Substituting f_2 and checking derivatives gives

$$g_3^x = \max_{i \in [0,1]} \left\{ -i^2\beta/2 + i(\beta - \alpha + \delta) + \alpha \right\}, \quad j = \frac{\beta - \alpha + \delta}{\beta}$$
$$g_3^x = \frac{(\beta - \alpha + \delta)^2}{2\beta} + \alpha$$
$$g_3^r = \gamma + \max_{i \in [0,1]} \left\{ -i^2\beta/2 + i(\beta - \alpha) + \alpha \right\}, \quad k = \frac{\beta - \alpha}{\beta}$$
$$g_3^r = \frac{(\beta - \alpha)^2}{2\beta} + \alpha + \gamma$$

Then

$$f_3(i) \;=\; (i - i^2/2)\beta + i g_3^x + (1 - i) g_3^r \;=\; f_2(i) + i(\alpha + g_3^x - g_3^r) + (g_3^r - \alpha) \;\;.$$

Since α was specifically chosen to solve

$$\alpha = g_3^r - g_3^x = \gamma - \frac{2(\beta - \alpha)\delta + \delta^2}{2\beta} \;,$$

and with $c = g_3^r - \alpha$, we have our result for the base case of $n = 3$. The general case follows by induction: if we assume that $f_{n-1}(i) = f_2(i) + (n - 3)c$, then clearly $g_n^x = g_3^x + (n - 3)c$ and $g_n^r = g_3^r + (n - 3)c$, and then

$$\begin{aligned} f_n(i) &= (i - i^2/2)\beta + i g_n^x + (1 - i) g_n^r \\ &= f_2(i) + i(\alpha + g_3^x - g_3^r) + (n - 3)c + (g_3^r - \alpha) \\ &= f_2(i) + (n - 2)c \;\;, \end{aligned}$$

which proves our lemma. $\square$

As before, $\dim_f(P(T_n))$ is bounded below by $w_n/\max_i\{f_n(i)\}$. (More precisely, the bound will approach this value as $q \to \infty$.) But $w_n = w_2 + (\beta+\delta+\gamma)n$ and $f_n(i) = f_2(i) + cn$. So as $n \to \infty$, the limiting value of this lower bound is just $(\beta+\delta+\gamma)/c$. Our goal now becomes to choose the values of β, δ and γ which maximize this quantity. More specifically, we must solve the following non-linear optimization problem:

$$\max_{\alpha,\beta,\delta,\gamma} \frac{\beta+\delta+\gamma}{c} \text{ s.t. } c = \frac{(\beta-\alpha)^2}{2\beta} + \gamma, \quad \alpha = \frac{\beta\gamma - \beta\delta + \delta^2/2}{\beta - \delta}, \quad \gamma \geq \delta.$$

Although we cannot hope for a clean analytic solution, we may reduce the problem somewhat. Notice that our objective function is not affected if we proportionally scale each of β, δ and γ, since c and α would also be scaled by the same proportion. Thus we may arbitrarily choose $\beta = 1$, giving

$$\alpha = \frac{\gamma - \delta + \delta^2/2}{1-\delta}$$

$$c = g_3^r - \alpha = (1-\alpha)^2/2 + \gamma = \frac{1}{2}\left(\frac{1-\gamma-\delta^2/2}{1-\delta}\right)^2 + \gamma \ .$$

Substituting these into our objective function gives

$$\max_{0\leq\delta\leq\gamma} \frac{2(1-\delta)^2(1+\delta+\gamma)}{\delta^4/4 + \delta^2\gamma + \delta^2 + \gamma^2 - 4\delta\gamma + 1}$$

The value of this optimization problem will be the best possible value of $w_n/\max_i\{f_n(i)\}$, and will thus be a lower bound on $\dim_f(P(T_n))$ as $n, q \to \infty$.

At this time, we are still attempting an analytic proof that the solution to this optimization problem is the same as the upper bound z_0. We have, however, used Mathematica to determine that the two quantities are equal out to 2000 decimal places of accuracy, so there can be little doubt that they are, in fact, the same number. □

4.6 Infinite Trees

We have already established that 2.44504 is a tight upper bound on the fractional dimension of posets of infinite trees so long as their maximum degree is bounded. What if an infinite tree is of unbounded maximum degree? The answer to this mystery is as close as the stars.

The following is a Corollary of a result that was discovered by Tom Trotter and John Moore, but never published.

Lemma 4.8 *Any tree T has* $\dim(P(T)) \leq 3$.

Proof. In other words, any poset $P(T)$ of a tree T has a size three realizer. We already know this result to be true when T is finite, since finite trees are planar and Schnyder [12] showed that $\dim(P(G)) \leq 3$ when G is planar. The following proof works equally well for finite and infinite trees.

We must specify the three linear extensions in our proposed realizer, and then check that any critical pair of types (i)-(v) gets reversed by at least one of these. For any tree T, draw T rooted, and impose a left-to-right ordering on each vertex's children. We then create the order of $V(T)$ in L_1 based on a left-to-right depth-first search, but we add a vertex only at the *last* time the search passes through it. Therefore a vertex comes below all its descendants in the ordering of L_1. In other words, for $u, v \in V(T)$, we have $v > u$ in

L_1 if either **(a)** v is a descendant of u in T, or **(b)** if x is u and v's lowest common ancestor in T, and y_u and y_v are x's children which contain u and v, respectively, in their subtrees, then y_v is left of y_u in the ordering of x's children within T. L_2 is constructed similarly, except that we apply a right-to-left depth-first search. L_3 is simply a top-down ordering, where the root is first, followed by all the root's children, etc.; the order of vertices from within any given level of T is unimportant.

We now check that $\{L_1, L_2, L_3\}$ is a realizer of $P(T)$ by checking that any critical pair of types (i)-(v) gets reversed by at least one of these linear extensions. Recall that, in order to reverse (u, vw), we must have $u > v, w$ in L_i.

(i) For critical pair (u, vw), v and w are descendants of u in T, and so $u > v, w$ in L_3.

(ii) For critical pair (u, vw), v and w are ancestors of u, and so are recorded *after* u in either of the searches defining L_1 and L_2; that is, $u > v, w$.

(iii) For critical pair (u, vw), let y_u be the child of v whose subtree contains u ($y_u \neq w$). If w is right of y_u in the ordering of v's children within T, then $u \geq y_u > w > v$ in L_1; otherwise, w is left of y_u, and we see this ordering within L_2.

(iv) For critical pair (u, vw), let x be u and v's lowest common ancestor, and y_u and y_v be the children of x whose subtrees contain u and vw, respectively. If y_v is right of y_u in the ordering of x's children within T, then $u \geq y_u > w > v \geq y_v$ in L_1; otherwise, y_v is left of y_u, and we see this ordering within L_2.

(v) For critical pair (branch,leaf), we always see branch>leaf in L_3.

So $\{L_1, L_2, L_3\}$ reverses every critical pair of $P(T)$, and we're done. □

Theorem 4.9 *If T is an infinite tree of unbounded degree, then* $\dim_f(P(T)) = 3$.

Proof. The previous Lemma establishes 3 as an upper bound. Now, another way of saying that T has unbounded degree is to say that, for every positive integer n, T has the n-star S_n as a subgraph. Suppose this is true for T. For fixed positive integer b, let $\dim_b(P(T)) = d$. Let $n = 2^d + 1$, and consider $S_n \subset T$ and any smallest b-fold realizer $\mathcal{R}$ of $P(S_n)$. In each of the d linear extensions in $\mathcal{R}$, a leaf of S_n is either above or below the root (center), so for all of $\mathcal{R}$, we may describe this vertex's position relative to the root by a length d binary sequence (1=above root, 0=below root). There are only 2^d possible sequences, but $2^d + 1$ leaves. So by the pigeonhole principle, there must be two leaves u and v that have the same position relative to the root in every linear extension in $\mathcal{R}$. Well, (r, u), (r, v), (u, rv) and (v, ru) are all critical pairs that must each be reversed b times by $\mathcal{R}$. To reverse (r, u) and (r, v), we must have b linear extensions where u and v are below r. To reverse (u, rv), we must have b linear extensions where $u > v > r$, and b more with $v > u > r$ to reverse (v, ru). These three sets of b linear extensions are necessarily disjoint, so we must have $d \geq 3b$. Thus

$$\dim_f(P(T)) = \lim_{b\to\infty} \frac{\dim_b(P(T))}{b} \geq \lim_{b\to\infty} \frac{\dim_b(P(S_n))}{b} \geq \lim_{b\to\infty} \frac{3b}{b} = 3 \ .$$

□

Thus there is a gap in fractional dimension between posets of infinite trees with bounded and unbounded degree: the former is bounded above by $z_0 \doteq 2.44504$, while the later is always 3. In particular, note that, while $\dim_f$ of the poset of an infinite star is 3, $\dim_f$ of the poset of any finite subgraph (finite star) is bounded above by $1 + \sqrt{2}$. This result is analogous to that of χ_f versus $\overline{\chi_f}$ from Chapter 2, and in fact, the graph $G^2_{1,1}$ (for which

$\chi_f = 3$ and $\overline{\chi_f} = 1 + \sqrt{2}$) was originally constructed based on this observation about posets.

5 Summary of Results and Open Problems

Chapter 2

In Chapter 2, for an infinite graph G, we defined $\overline{\chi_f}(G)$ to be the supremum of χ_f of all of G's finite subgraphs. Our results answer an open problem posed by Leader [8], and are as follows:

- For an infinite graph G, $\omega_f(G) = \overline{\chi_f}(G)$.
- For the class of graphs $G^n_{r,s}$ with $r, s \in Q_f$ and integer $n \geq 2$, $\chi_f(G^n_{r,s}) = r + ns$ while $\overline{\chi_f}(G^n_{r,s}) = r/p_0$, where p_0 is the unique real root of $rx^n + nsx - r = 0$ in $(0, 1)$.
- The above class of graphs not only has the property that $\overline{\chi_f} < \chi_f < \infty$, but along with the preceding result, demonstrates that ω_f and χ_f aren't necessarily equal for infinite graphs.

While the class of graphs $G^n_{r,s}$ and several simple extensions of it cover many possible values of the ordered pair $(\overline{\chi_f}, \chi_f)$, there are potentially many such values that are not achieved.

- ◦ Given any $y > x > 2$, does there exist an infinite graph G for which $(\overline{\chi_f}(G), \chi_f(G)) = (x, y)$?
- ◦ The class of perfect graphs (for which $\omega(G) = \chi(G)$) is well studied. We could define the class of *fractionally perfect* infinite graphs to be those with $\omega_f(G) =$

$\chi_f(G)$. Which, of course, poses the problem: Characterize the class of fractionally perfect infinite graphs.

Chapter 3

In Chapter 3, we took $n \xrightarrow{f} (x, y)$ to mean that, if $K_n = H_1 \oplus H_2$, then $\omega_f(H_1) \geq x$ or $\omega_f(H_2) \geq y$. We define $r_f(x, y)$ to be the least integer for which this is true, and similarly define $r_f(x_1, \ldots, x_p)$ as the extension from 2 to p colors. We then prove

- Let $x = k + \varepsilon$ and $y = l + \delta$ for integers $k, l \geq 1$ and $0 < \varepsilon, \delta \leq 1$, and let $q = \min\{\lceil \varepsilon l \rceil, \lceil \delta k \rceil\}$. Then $r_f(x, y) = kl+q$. Contrast this with the intractability of exact value calculations and exponential growth rate of the ordinary Ramsey numbers.

- Given $x_1, \ldots, x_p \geq 2$, we have the following recursive bound:

$$r_f(x_1, \ldots, x_p) \leq \lceil (r_f(x_1, \ldots x_{p-1}) - 1)x_p \rceil.$$

  We present several special cases where this upper bound actually gives the correct value of r_f, most notably the previous case of $p = 2$ and the case where all x_i are the same integer.

- We also briefly examine several similar generalizations of Ramsey numbers, including b-fold Ramsey numbers and Lovász-ϑ Ramsey numbers.

The principle open problems are as follows:

◦ Prove or disprove Conjecture 3.4 that the recursive upper bound on $r_f(x_1, \ldots, x - p)$ *always* gives the correct value.

- There is much work that remains to be done on the subject of b-Ramsey numbers. In particular, $r_b(x, y)$ is, roughly, an increasing function of x and y (presumably exhibiting exponential growth) and a decreasing function of b (which almost everywhere approaches $r_f(x, y)$ pointwise from above, even though $r_f(x, y)$ only increases linearly in x or y).

- Although we established that $r_\vartheta(x, y)$ is very near xy, it remians to determine an exact formula for this quantity as we did for $r_f(x, y)$.

Chapter 4

In Chapter 4, we examined the fractional dimension of posets of trees, and found the following:

- $1 + \sqrt{2}$ is the best possible upper bound on $\dim_f$ of posets of finite stars[31].

- 7/3 is the best possible upper bound on $\dim_f$ of posets of (finite or infinite) binary trees.

- $z_0 \doteq 2.44504$ (a root of $z^3 - 7z^2 + 14z - 7 = 0$) is the best possible upper bound on $\dim_f$ of posets of trees (finite or infinite) with bounded maximum degree.

- For any infinite tree T with unbounded maximum degree, $\dim_f(P(T)) = 3$.

We still have the following open problems:

[31] This result was previously observed by Brightwell and Scheinerman [2].

- While $\frac{5q-3}{2q-1}$ is the best *known* upper bound on $\dim_f(P(T))$ for the class of q-ary trees for $q = 2, 3, 4, 5$, it is not, in general, best possible. What *is* the best-possible upper bound as a function of q? We do know that for $q = 2$ the value $7/3$ is correct, and that as $q \to \infty$ the value approaches 2.44504.

- Brightwell and Scheinerman [2] proved that posets of finite graphs have $\dim_f = 3$ iff the graph contains a 3-cycle. And $\dim_f$ of posets of trees is covered herein. This leaves the class of graphs with smallest cycle larger than size 3 unexplored. What can be said about $\dim_f$ of the posets of these graphs?

A Appendix: Leftovers

A.1 χ_f of Lexicographic Products

Scheinerman and Ullman [11] present a proof that $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$ for finite graphs by proving "$\geq$" and "$\leq$", but one direction makes use of fractional clique number and the fact that $\chi_f = \omega_f$ for finite graphs. Since Chapter 2 shows us that this is not the case for infinite graphs, a new proof is needed in this case. The other direction only requires minor modification to accommodate infinite graphs, but we present both for completeness. We use the fractional coloring notation developed in Chapter 2.

Lemma A.1 *For any two graphs G and H, finite or infinite, $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$.*

Proof. First we need some additional notation. Given graphs G and H, let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be the sets of independent sets of G, H and $G[H]$, respectively. Recall that, roughly, $G[H]$ is constructed by replacing each vertex of G with a copy of H. Then for $u \in V(G)$, let $u[H]$ be the copy of H put at u. For any $K \in \mathcal{K}$, let $K|_G = \{u \in V(G) : u[H] \cap K \neq \emptyset\}$, and note that $K|_G \in \mathcal{I}$. Finally, note that, for $I \in \mathcal{I}$ and $J \in \mathcal{J}$, we have $I \times J \in \mathcal{K}$.

Let f_G and f_H be fractional colorings of G and H, respectively. Then we may define the function $f = f_G \times f_H : \mathcal{K} \to [0,1]$ by $f(I \times J) = f_G(I)f_H(J)$ and $f(K) = 0$ if $K \in \mathcal{K}$ cannot be written as any $I \times J$. Then for any $(u,v) \in V(G[H])$, we have

$$\sum_{K \in \mathcal{K} : (u,v) \in K} f(K) = \sum_{I \times J \in \mathcal{K} : (u,v) \in I \times J} f(I \times J)$$

$$
\begin{aligned}
&= \sum_{I\in\mathcal{I}:u\in I}\left(\sum_{J\in\mathcal{J}:v\in J} f_G(I)f_H(J)\right)\\
&= \left(\sum_{I\in\mathcal{I}:u\in I} f_G(I)\right)\cdot\left(\sum_{J\in\mathcal{J}:v\in J} f_H(J)\right)\\
&\geq 1\cdot 1 = 1,
\end{aligned}
$$

and so f is itself a valid fractional coloring of $G[H]$. Equations similar to the above with the sums taken over *all* $I \times J \in \mathcal{K}$ show that $w(f) = w(f_G)w(f_H)$. Since we can pick f_G and f_H with values arbitrarily close to $\chi_f(G)$ and $\chi_f(H)$, respectively, we can construct fractional colorings of $G[H]$ with values arbitrarily close to $\chi_f(G)\chi_f(H)$, and so $\chi_f(G[H]) \leq \chi_f(G)\chi_f(H)$.

Next, suppose that f is a fractional coloring of $G[H]$ with value $w(f) < \chi_f(G)\chi_f(H)$. For $u \in V(G)$, let $w_u(f) = \sum f(K)$, summed over all $K \in \mathcal{K}$ which intersect $u[H]$, so that f puts total weight $w_u(f)$ on $u[H]$. Now, $u[H]$ is a copy of H, and f restricted to the independent sets intersecting $u[H]$ is still a fractional coloring of $u[H]$, so we must have $w_u(f) \geq \chi_f(H)$ for all $u \in V(G)$. Next, if we collapse each $u[H]$ into a single vertex u (giving us G), any $K \in \mathcal{K}$ collapses to some $I \in \mathcal{I}$; specifically, K becomes $K|_G$. We set

$$f_G(I) = \frac{1}{\chi_f(H)} \sum_{K\in\mathcal{K}:K|_G=I} f(K|_G) \quad ,$$

so that $w(f_G) = w(f)/\chi_f(H)$. Since f put weight at least $\chi_f(H)$ on each $u[H]$, f_G must put weight at least 1 on each $u \in V(G)$, and so is a valid fractional coloring of G with weight

$$\frac{w(f)}{\chi_f(H)} < \frac{\chi_f(G)\chi_f(H)}{\chi_f(H)} = \chi_f(G).$$

This is a contradiction. Thus no fractional coloring of $G[H]$ can have weight less than $\chi_f(G)\chi_f(H)$, and we are done. □

A.2 $\omega_f(G) = \lim_{b\to\infty} \frac{\omega_b(G)}{b}$ for Infinite Graphs

In fact, the following Theorem and proof also apply to finite graphs, but the result is already known in that case (see [11]).

So that we may discuss $\omega_b(G)$ of infinite graphs, we use its formulation as the size of the largest multiset of $V(G)$ with the property that no independent set of G contains more than b elements from this multiset (counting repetition). A b-fold clique is *any* multiset of $V(G)$ which satisfies this property.

Theorem A.2 *For any finite or infinite graph G,*

$$\lim_{b\to\infty} \frac{\omega_b(G)}{b} = \omega_f(G).$$

Proof. We take ω_f as defined in Chapter 2. Analogously to Lemma 1.1 and Theorem 2.1, we may divide any b-fold clique by b to get a fractional clique. However, since ω_f is formulated as a maximization problem, $\omega \leq \omega_b/b \leq \omega_f$, and so $\lim_{b\to\infty} \frac{\omega_b(G)}{b} \leq \omega_f(G)$.

The other inequality also proceeds much like the proof of Theorem 2.1, but with fewer complications. For any (finite or infinite) graph G, fix $\varepsilon > 0$. Then let g be a fractional clique with $\omega_f(G) - \varepsilon/3 \leq w(g) \leq \omega_f(G)$. Since $w(g)$ is the (possibly infinite) sum of the weights put on vertices by g, there are finite partial sums with values arbitrarily close to $w(g)$. In particular, choose a finite set $S \subseteq V(G)$ such that $\sum_{v\in S} g(v) \geq w(g) - \varepsilon/3 \geq \omega_f(G) - 2\varepsilon/3$, and let $g' : V(G) \to [0,1]$ equal g on the set S and be zero elsewhere, so that $w(g') = \sum_{v \in S} g(v)$. Since g' can't put more weight on the vertices of an independent set than g, it is also a fractional clique. Finally, take $b \geq (3|S|)/\varepsilon$, and create g'' by rounding all values $g'(v)$ down to the nearest multiple of $1/b$. This removes total weight at most $|S|/b \leq \varepsilon/3$ from g', and still leaves g'' a valid fractional clique, which has weight $w(g'') \geq w(g') - \varepsilon/3 \geq \omega_f(G) - \varepsilon$. We may now define the b-fold clique g_b to be the multiset of $V(G)$ where each vertex v is included $b \cdot g''(v)$ times. This multiset is a valid b-fold clique since no independent set can contain more than b of these elements (counting repetition). So the weight (size) of g_b is a lower bound on $\omega_b(G)$, and

$$\frac{\omega_b(G)}{b} \geq \frac{|g_b|}{b} = w(g'') \geq \omega_f(G) - \varepsilon.$$

Since this is true of *any* $b \geq (3|S|)/\varepsilon$, it is clear that $\omega_b(G)/b$ actually approaches $\omega_f(G)$ in the limit. □

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Vita

Gregory Matthew Levin was born on July 2nd, 1970 in Inglewood, CA to Michael and Tommy Kay Levin. Even though he spent his first 18 years in Hermosa Beach, he never learned to surf. He attended Hermosa View Elementary School, Hermosa Valley Middle School, and Redondo Union High School, where he was valedictorian. An amusing "Junk Mail Flyer" led him to Harvey Mudd College in Claremont, CA, where he learned to juggle (but not very well), play Tron (exceedingly well) and manage student arts publications (well, sort of manage). He also developed an interest in Discrete Math and Optimization working under his advisor, Prof. Arthur Benjamin. In May of 1992, he received his Bachelor's of Science in Mathematics, graduating with distinction. He spent the next five years in the Mathematical Sciences department at The Johns Hopkins University, where he received the Abel Wolman and Rufus Isaacs Graduate Fellowships. His advisor, Prof. Edward Scheinerman, introduced him to the field of Fractional Graph Theory, which became the focus of his graduate research. The material in this dissertation was also prepared for three journal papers:

- (with M. Jacobson and E. Scheinerman) On Fractional Ramsey Numbers, *Discrete Mathematics*, to appear.
- The Fractional Chromatic Gap of Infinite Graphs, submitted.
- The Fractional Dimension of Posets of Trees, in preparation.

He also presented some of this work at The 1995 Southeastern Conference on Combinatorics, Graph Theory and Computing. He successfully defended this dissertation on June

5, 1997, and in so doing completed his Ph.D. in Mathematical Sciences. He is currently living in fear of having to get a real job.