5 Summary of Results and Open Problems

Chapter 2

In Chapter 2, for an infinite graph G, we defined $\overline{\chi_f}(G)$ to be the supremum of χ_f of all of G's finite subgraphs. Our results answer an open problem posed by Leader [8], and are as follows:

- For an infinite graph G, $\omega_f(G) = \overline{\chi_f}(G)$.
- For the class of graphs $G^n_{r,s}$ with $r, s \in Q_f$ and integer $n \geq 2$, $\chi_f(G^n_{r,s}) = r + ns$ while $\overline{\chi_f}(G^n_{r,s}) = r/p_0$, where p_0 is the unique real root of $rx^n + nsx - r = 0$ in $(0, 1)$.
- The above class of graphs not only has the property that $\overline{\chi_f} < \chi_f < \infty$, but along with the preceding result, demonstrates that ω_f and χ_f aren't necessarily equal for infinite graphs.

While the class of graphs $G^n_{r,s}$ and several simple extensions of it cover many possible values of the ordered pair $(\overline{\chi_f}, \chi_f)$, there are potentially many such values that are not achieved.

- ○ Given any $y > x > 2$, does there exist an infinite graph G for which $(\overline{\chi_f}(G), \chi_f(G)) = (x, y)$?
- ○ The class of perfect graphs (for which $\omega(G) = \chi(G)$) is well studied. We could define the class of *fractionally perfect* infinite graphs to be those with $\omega_f(G) =$

$\chi_f(G)$. Which, of course, poses the problem: Characterize the class of fractionally perfect infinite graphs.

Chapter 3

In Chapter 3, we took $n \xrightarrow{f} (x,y)$ to mean that, if $K_n = H_1 \oplus H_2$, then $\omega_f(H_1) \geq x$ or $\omega_f(H_2) \geq y$. We define $r_f(x,y)$ to be the least integer for which this is true, and similarly define $r_f(x_1,\ldots,x_p)$ as the extension from 2 to p colors. We then prove

- Let $x = k+\varepsilon$ and $y = l+\delta$ for integers $k,l \geq 1$ and $0 < \varepsilon,\delta \leq 1$, and let $q = \min\{\lceil \varepsilon l \rceil, \lceil \delta k \rceil\}$. Then $r_f(x,y) = kl+q$. Contrast this with the intractability of exact value calculations and exponential growth rate of the ordinary Ramsey numbers.

- Given $x_1,\ldots,x_p \geq 2$, we have the following recursive bound:

  $$r_f(x_1,\ldots,x_p) \leq \lceil (r_f(x_1,\ldots x_{p-1}) - 1)x_p \rceil.$$

  We present several special cases where this upper bound actually gives the correct value of r_f, most notably the previous case of $p = 2$ and the case where all x_i are the same integer.

- We also briefly examine several similar generalizations of Ramsey numbers, including b-fold Ramsey numbers and Lovász-ϑ Ramsey numbers.

The principle open problems are as follows:

- ○ Prove or disprove Conjecture 3.4 that the recursive upper bound on $r_f(x_1,\ldots,x-p)$ *always* gives the correct value.

- ○ There is much work that remains to be done on the subject of b-Ramsey numbers. In particular, $r_b(x, y)$ is, roughly, an increasing function of x and y (presumably exhibiting exponential growth) and a decreasing function of b (which almost everywhere approaches $r_f(x, y)$ pointwise from above, even though $r_f(x, y)$ only increases linearly in x or y).
- ○ Although we established that $r_\vartheta(x, y)$ is very near xy, it remians to determine an exact formula for this quantity as we did for $r_f(x, y)$.

Chapter 4

In Chapter 4, we examined the fractional dimension of posets of trees, and found the following:

- $1 + \sqrt{2}$ is the best possible upper bound on $\dim_f$ of posets of finite stars[31].
- 7/3 is the best possible upper bound on $\dim_f$ of posets of (finite or infinite) binary trees.
- $z_0 \doteq 2.44504$ (a root of $z^3 - 7z^2 + 14z - 7 = 0$) is the best possible upper bound on $\dim_f$ of posets of trees (finite or infinite) with bounded maximum degree.
- For any infinite tree T with unbounded maximum degree, $\dim_f(P(T)) = 3$.

We still have the following open problems:

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[31] This result was previously observed by Brightwell and Scheinerman [2].

- While $\frac{5q-3}{2q-1}$ is the best *known* upper bound on $\dim_f(P(T))$ for the class of q-ary trees for $q = 2, 3, 4, 5$, it is not, in general, best possible. What *is* the best-possible upper bound as a function of q? We do know that for $q = 2$ the value $7/3$ is correct, and that as $q \to \infty$ the value approaches 2.44504.
- Brightwell and Scheinerman [2] proved that posets of finite graphs have $\dim_f = 3$ iff the graph contains a 3-cycle. And $\dim_f$ of posets of trees is covered herein. This leaves the class of graphs with smallest cycle larger than size 3 unexplored. What can be said about $\dim_f$ of the posets of these graphs?

A Appendix: Leftovers

A.1 χ_f of Lexicographic Products

Scheinerman and Ullman [11] present a proof that $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$ for finite graphs by proving "$\geq$" and "$\leq$", but one direction makes use of fractional clique number and the fact that $\chi_f = \omega_f$ for finite graphs. Since Chapter 2 shows us that this is not the case for infinite graphs, a new proof is needed in this case. The other direction only requires minor modification to accommodate infinite graphs, but we present both for completeness. We use the fractional coloring notation developed in Chapter 2.

Lemma A.1 *For any two graphs G and H, finite or infinite, $\chi_f(G[H]) = \chi_f(G)\chi_f(H)$.*

Proof. First we need some additional notation. Given graphs G and H, let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be the sets of independent sets of G, H and $G[H]$, respectively. Recall that, roughly, $G[H]$ is constructed by replacing each vertex of G with a copy of H. Then for $u \in V(G)$, let $u[H]$ be the copy of H put at u. For any $K \in \mathcal{K}$, let $K|_G = \{u \in V(G) : u[H] \cap K \neq \emptyset\}$, and note that $K|_G \in \mathcal{I}$. Finally, note that, for $I \in \mathcal{I}$ and $J \in \mathcal{J}$, we have $I \times J \in \mathcal{K}$.

Let f_G and f_H be fractional colorings of G and H, respectively. Then we may define the function $f = f_G \times f_H : \mathcal{K} \to [0,1]$ by $f(I \times J) = f_G(I)f_H(J)$ and $f(K) = 0$ if $K \in \mathcal{K}$ cannot be written as any $I \times J$. Then for any $(u,v) \in V(G[H])$, we have

$$\sum_{K \in \mathcal{K} : (u,v) \in K} f(K) \quad = \sum_{I \times J \in \mathcal{K} : (u,v) \in I \times J} f(I \times J)$$

$$
\begin{aligned}
&= \sum_{I\in\mathcal{I}:u\in I}\left(\sum_{J\in\mathcal{J}:v\in J} f_G(I)f_H(J)\right)\\
&= \left(\sum_{I\in\mathcal{I}:u\in I} f_G(I)\right)\cdot\left(\sum_{J\in\mathcal{J}:v\in J} f_H(J)\right)\\
&\geq 1\cdot 1 = 1,
\end{aligned}
$$

and so f is itself a valid fractional coloring of $G[H]$. Equations similar to the above with the sums taken over *all* $I \times J \in \mathcal{K}$ show that $w(f) = w(f_G)w(f_H)$. Since we can pick f_G and f_H with values arbitrarily close to $\chi_f(G)$ and $\chi_f(H)$, respectively, we can construct fractional colorings of $G[H]$ with values arbitrarily close to $\chi_f(G)\chi_f(H)$, and so $\chi_f(G[H]) \leq \chi_f(G)\chi_f(H)$.

Next, suppose that f is a fractional coloring of $G[H]$ with value $w(f) < \chi_f(G)\chi_f(H)$. For $u \in V(G)$, let $w_u(f) = \sum f(K)$, summed over all $K \in \mathcal{K}$ which intersect $u[H]$, so that f puts total weight $w_u(f)$ on $u[H]$. Now, $u[H]$ is a copy of H, and f restricted to the independent sets intersecting $u[H]$ is still a fractional coloring of $u[H]$, so we must have $w_u(f) \geq \chi_f(H)$ for all $u \in V(G)$. Next, if we collapse each $u[H]$ into a single vertex u (giving us G), any $K \in \mathcal{K}$ collapses to some $I \in \mathcal{I}$; specifically, K becomes $K|_G$. We set

$$
f_G(I) = \frac{1}{\chi_f(H)} \sum_{K\in\mathcal{K}:K|_G=I} f(K|_G)\ \ ,
$$

so that $w(f_G) = w(f)/\chi_f(H)$. Since f put weight at least $\chi_f(H)$ on each $u[H]$, f_G must put weight at least 1 on each $u \in V(G)$, and so is a valid fractional coloring of G with weight

$$
\frac{w(f)}{\chi_f(H)} < \frac{\chi_f(G)\chi_f(H)}{\chi_f(H)} = \chi_f(G).
$$

This is a contradiction. Thus no fractional coloring of $G[H]$ can have weight less than $\chi_f(G)\chi_f(H)$, and we are done. $\square$

A.2 $\omega_f(G) = \lim_{b\to\infty} \frac{\omega_b(G)}{b}$ for Infinite Graphs

In fact, the following Theorem and proof also apply to finite graphs, but the result is already known in that case (see [11]).

So that we may discuss $\omega_b(G)$ of infinite graphs, we use its formulation as the size of the largest multiset of $V(G)$ with the property that no independent set of G contains more than b elements from this multiset (counting repetition). A b-fold clique is *any* multiset of $V(G)$ which satisfies this property.

Theorem A.2 *For any finite or infinite graph G,*

$$\lim_{b\to\infty} \frac{\omega_b(G)}{b} = \omega_f(G).$$

Proof. We take ω_f as defined in Chapter 2. Analogously to Lemma 1.1 and Theorem 2.1, we may divide any b-fold clique by b to get a fractional clique. However, since ω_f is formulated as a maximization problem, $\omega \leq \omega_b/b \leq \omega_f$, and so $\lim_{b\to\infty} \frac{\omega_b(G)}{b} \leq \omega_f(G)$.

The other inequality also proceeds much like the proof of Theorem 2.1, but with fewer complications. For any (finite or infinite) graph G, fix $\varepsilon > 0$. Then let g be a fractional clique with $\omega_f(G) - \varepsilon/3 \leq w(g) \leq \omega_f(G)$. Since $w(g)$ is the (possibly infinite) sum of the weights put on vertices by g, there are finite partial sums with values arbitrarily close to $w(g)$. In particular, choose a finite set $S \subseteq V(G)$ such that $\sum_{v\in S} g(v) \geq w(g) - \varepsilon/3 \geq \omega_f(G) - 2\varepsilon/3$, and let $g' : V(G) \to [0,1]$ equal g on the set S and be zero elsewhere, so that $w(g') = \sum_{v \in S} g(v)$. Since g' can't put more weight on the vertices of an independent set than g, it is also a fractional clique. Finally, take $b \geq (3|S|)/\varepsilon$, and create g'' by rounding all values $g'(v)$ down to the nearest multiple of $1/b$. This removes total weight at most $|S|/b \leq \varepsilon/3$ from g', and still leaves g'' a valid fractional clique, which has weight $w(g'') \geq w(g') - \varepsilon/3 \geq \omega_f(G) - \varepsilon$. We may now define the b-fold clique g_b to be the multiset of $V(G)$ where each vertex v is included $b \cdot g''(v)$ times. This multiset is a valid b-fold clique since no independent set can contain more than b of these elements (counting repetition). So the weight (size) of g_b is a lower bound on $\omega_b(G)$, and

$$\frac{\omega_b(G)}{b} \geq \frac{|g_b|}{b} = w(g'') \geq \omega_f(G) - \varepsilon.$$

Since this is true of *any* $b \geq (3|S|)/\varepsilon$, it is clear that $\omega_b(G)/b$ actually approaches $\omega_f(G)$ in the limit. □

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Vita

Gregory Matthew Levin was born on July 2nd, 1970 in Inglewood, CA to Michael and Tommy Kay Levin. Even though he spent his first 18 years in Hermosa Beach, he never learned to surf. He attended Hermosa View Elementary School, Hermosa Valley Middle School, and Redondo Union High School, where he was valedictorian. An amusing "Junk Mail Flyer" led him to Harvey Mudd College in Claremont, CA, where he learned to juggle (but not very well), play Tron (exceedingly well) and manage student arts publications (well, sort of manage). He also developed an interest in Discrete Math and Optimization working under his advisor, Prof. Arthur Benjamin. In May of 1992, he received his Bachelor's of Science in Mathematics, graduating with distinction. He spent the next five years in the Mathematical Sciences department at The Johns Hopkins University, where he received the Abel Wolman and Rufus Isaacs Graduate Fellowships. His advisor, Prof. Edward Scheinerman, introduced him to the field of Fractional Graph Theory, which became the focus of his graduate research. The material in this dissertation was also prepared for three journal papers:

- (with M. Jacobson and E. Scheinerman) On Fractional Ramsey Numbers, *Discrete Mathematics,* to appear.
- The Fractional Chromatic Gap of Infinite Graphs, submitted.
- The Fractional Dimension of Posets of Trees, in preparation.

He also presented some of this work at The 1995 Southeastern Conference on Combinatorics, Graph Theory and Computing. He successfully defended this dissertation on June

5, 1997, and in so doing completed his Ph.D. in Mathematical Sciences. He is currently living in fear of having to get a real job.