

Analysis of Flow Past a Cylinder Attached to a Spring

Michael Franklin

May 1, 2008

1 Introduction

The flow of fluid past a cylinder has been widely studied both analytically and numerically, and provides insight to the behavior of flow given different fluid properties. Particularly interesting is the transition from steady flow to vortex shedding in the wake of the cylinder. This behavior is dependent on several parameters, but most importantly, the Reynolds number (Re), which is a dimensionless measure of the range of scales upon which different flow behavior is occurring. For example, laminar flow has large-scale flow behavior, corresponding to low Re ($Re < Re_c$), while turbulent flow has both large and small scale flow behavior, corresponding to high Re ($Re > Re_c$) [9]. The transition from steady flow to vortex shedding happens at a critical value Re_c , which is in the range of 46-49 in the literature [3, 4, 5, 9].

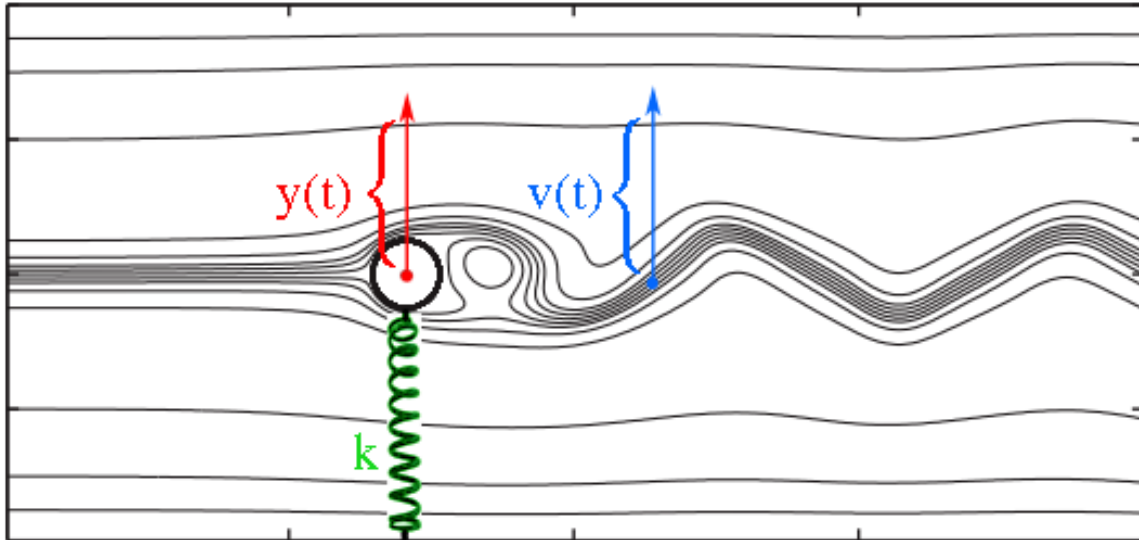


Figure 1: Schematic of the model: fluid flow past a cylinder attached to a spring. This represents a coupling between the Stuart-Landau equation describing the transverse velocity in the wake, $v(t)$, and damped driven harmonic oscillation of the displacement $y(t)$.

A mathematical model which describes the behavior of the flow past a cylinder is the complex Stuart-Landau equation. This model is derived by perturbations of the linearized Navier Stokes

equations, and will be described further in the background section. Due to the accuracy of this model in describing the transverse velocity $v(t)$ in the wake of the cylinder, it can be coupled to other mathematical models to see the resulting behavior. This paper discusses the results when the Stuart-Landau equation is coupled to a damped driven harmonic oscillator equation, which describes the displacement of the cylinder $y(t)$, such that the transverse velocity $v(t)$ is the driving force in the oscillator equation. The schematic of the model is shown in Figure 1.

2 Background

Complex Stuart-Landau Equation

The complex Stuart-Landau equation has been used as a mathematical model to study the behavior of two-dimensional vortex shedding in the wake of a cylinder for low Reynolds numbers. The complex Stuart-Landau equation is derived by performing stability analysis of a steady flow past a cylinder. This is done by Provansal in [3] by introducing a non-stationary perturbation $u_1(x, y, t)$ of the steady state solution $u_0(x, y)$ of the Navier Stokes equation, which is expanded as

$$u_1(x, y, t) = \sum_{i=1} [A_i(t)g_i(x, y) + A_i^*(t)g_i(x, y)], \quad (1)$$

where $g_i(x, y)$ satisfies the boundary conditions, and $A_i(t)$ is the respective amplitude of the i^{th} oscillatory mode. It can be shown that the amplitude $A_i(t)$ satisfies the evolution equation

$$\frac{dA_i}{dt} = s_i A_i + G_i(A_j) \quad j = 1, 2, \dots, \quad (2)$$

where s_i is the linear coefficient, and G_i contains the nonlinear interaction of all modes, including the i^{th} mode, resulting from the nonlinear partial differential equation. The Stuart-Landau equation is then a truncated form of (2), given by Le Gal in [6] as

$$\frac{dA}{dt} = (a_R + ia_I)A - (\ell_R + i\ell_I)|A|^2A + \dots \quad (3)$$

where $A(t)$ is a complex-valued function of time t , and $a_R, a_I, \ell_R, \ell_I \in \mathbb{R}$, where the subscripts I, R represent imaginary and real coefficients, respectively. The Stuart-Landau equation (3) is typically truncated after the cubic term.

The function $A(t)$ can be thought of as a variety of physical measurable quantities, one being the lift coefficient experienced by the cylinder. In the context of modeling the flow in the wake of a cylinder, the oscillatory transverse velocity $v(t)$ is taken as the order parameter in this formulation. This velocity is shown in the schematic Figure 1 to run perpendicular to the incoming flow. Thus, truncating and plugging in the transverse velocity into (3) yields the following Stuart-Landau equation governing the transverse velocity in the wake of flow past a cylinder

$$\frac{dv}{dt} = (a_R + ia_I)v - (\ell_R + i\ell_I)|v|^2v \quad (4)$$

Since (4) is a complex valued function, the transverse velocity $v(t)$ can be written as

$$v(t) = \rho(t)e^{i\phi(t)}, \quad (5)$$

where $\rho(t) = |v(t)|$ is the real and non-negative amplitude, and $\phi(t)$ is the real phase of the transverse velocity $v(t)$. Plugging this into (4) gives the following system

$$\begin{aligned}\frac{d\rho}{dt} &= a_R\rho - \ell_R\rho^3, \\ \frac{d\phi}{dt} &= a_I - \ell_I\rho^2.\end{aligned}\tag{6}$$

where a_R controls the stability of the fixed point at the origin, a_I gives the frequency of infinitesimal oscillations, and ℓ_I determines the dependence of frequency on amplitude for larger amplitude oscillations [8]. We wish to analyze the behavior and stability of the system; to simplify the analysis, we will transform the system to Cartesian coordinates to construct the Jacobian matrix to determine the stability. Letting $x = \rho \cos(\phi)$ and $y = \rho \sin(\phi)$ and using (6), we arrive at

$$\begin{aligned}\dot{x} &= a_Rx - a_Iy + (\ell_Iy - \ell_Rx)(x^2 + y^2), \\ \dot{y} &= a_Ry + a_Ix - (\ell_Ix + \ell_Ry)(x^2 + y^2).\end{aligned}\tag{7}$$

It is clear that the system has a fixed point $(0, 0)$. Classifying the stability of this fixed point can be done by constructing the (Cartesian) Jacobian matrix associated with the system, and using (7) is given by

$$J|_{(0,0)} = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix}_{(0,0)} = \begin{pmatrix} a_R & -a_I \\ a_I & a_R \end{pmatrix}.\tag{8}$$

The eigenvalues of the Jacobian J evaluated at the fixed point $(0, 0)$ are given by the complex conjugate pair

$$\lambda = a_R \pm ia_I.\tag{9}$$

It is clear that as a_R increases from negative, through zero, to positive, both eigenvalues simultaneously cross over the imaginary axis in the complex plane. This behavior is indicative of a Hopf bifurcation with bifurcation parameter a_R , with a critical value of $a_R = 0$. In fact (6) is the normal form of a supercritical Hopf bifurcation [8].

The supercritical Hopf bifurcation can be described by analyzing the different ranges of the bifurcation parameter a_R . It is well established that the amplitude $|v|$ of oscillatory transverse velocity verifies Landau's law, given as

$$|v| \propto (Re - Re_c)^{1/2},\tag{10}$$

in which Re is the order parameter of the bifurcation and the exponent $1/2$ is known with an accuracy better than 1% [3]. In the transverse velocity formulation, the bifurcation parameter a_R can be written as

$$a_R = \frac{(Re - Re_c)\nu}{5d^2},\tag{11}$$

where d^2/ν is the viscous diffusion time which is characteristic of each cylinder [3]. Using this form of the bifurcation parameter, there are three ranges of the parameter to examine:

1. When $Re < Re_c$ the parameter $a_R < 0$ and the origin $(0, 0)$ is a stable spiral with exponential decay rate. This corresponds to laminar flow with disturbances decaying to zero.
2. When $Re = Re_c$ the parameter $a_R = 0$ and the origin $(0, 0)$ is a marginally stable spiral with algebraic (very slow) decay rate. This is the threshold of the vortex shedding regime.
3. When $Re > Re_c$ the parameter $a_R > 0$ and the origin $(0, 0)$ becomes unstable, and the solution settles down to a time-periodic limit cycle. This corresponds to periodic vortex shedding in the wake of the cylinder.

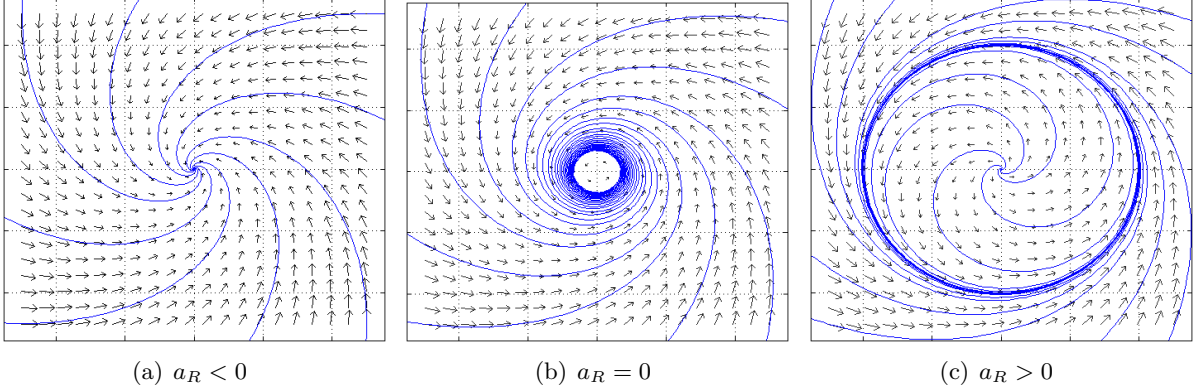


Figure 2: The phase plane for the Stuart-Landau system (6).

These different cases are shown in Figure 2. The most interesting case is when $a_R > 0$ and the limit cycle emerges, rendering the origin unstable. Finding the amplitude and angular velocity of the periodic limit cycle can be done by finding the nullcline of the amplitude equation in (6). When $\dot{\rho} = 0$ the amplitude is constant, which corresponds to a periodic limit cycle. Finding the nullcline of the equation for $\rho(t)$ yields

$$\frac{d\rho}{dt} = 0 \quad \text{if} \quad \rho(a_R - \ell_R \rho^2) = 0 \quad \Rightarrow \quad \rho^* = 0, \pm(a_R/\ell_R)^{1/2}, \quad (12)$$

where the trivial and the negative roots are unphysical. Thus, the periodic orbit settles to a limit cycle of amplitude

$$|v|_{LC} = \rho^* = (a_R/\ell_R)^{1/2}, \quad (13)$$

which is consistent with the Landau law (10). Plugging this into the equation for the phase yields the angular velocity $\omega_{LC} = \dot{\phi}$ of the trajectory around the limit cycle,

$$\omega_{LC} = \dot{\phi} = a_I - a_R \frac{\ell_I}{\ell_R}. \quad (14)$$

Thus, having the amplitude $|v|_{LC}$ and frequency ω_{LC} of the transverse velocity when the stable limit cycle is reached, we now have an explicit form of the stable limit cycle velocity $v_{LC}(t)$ given by

$$v_{LC}(t) = |v|_{LC} \exp(i\omega_{LC}t) = \left(\frac{a_R}{\ell_R}\right)^{1/2} \exp\left[\left(a_I - a_R \frac{\ell_I}{\ell_R}\right)t\right]. \quad (15)$$

Knowing this explicit form of the limit cycle velocity, we can use this to drive a harmonic oscillator by creating a “damping”-like forcing term. The Stuart-Landau model describes the transverse velocity of the flow globally with the same time evolution in the flow axis downstream and upstream of the cylinder, thus it is justified to assume that the velocity near the cylinder can act as a driving force [1]. The harmonic oscillation driven by this velocity is described in the following subsection.

Damped Driven Harmonic Oscillation

The general form of a damped driven harmonic oscillator is given by

$$\frac{d^2y}{dt^2} + 2c\frac{dy}{dt} + \omega_0^2y = F(t)/m, \quad (16)$$

where $y(t)$ is the displacement from equilibrium, $c = b/2m$ is proportional to the damping, $\omega_0^2 = k/m$ is the oscillator’s natural frequency, m is the mass of the cylinder, b is the damping coefficient, k is the spring constant, and $F(t)$ is the driving force of the system. In our case, we use the limit cycle velocity $v_{LC}(t)$ to construct a forcing term given by $F(t) = \beta v_{LC}(t)$, where $[\beta] = kg/s$.

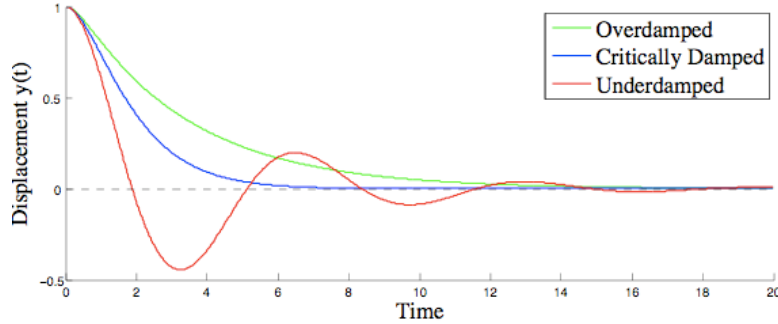


Figure 3: Different damping behavior of the homogeneous (non-driven) damped harmonic oscillator.

The homogeneous equation (i.e. setting $F(t) = 0$) gives insight to different damping regimes present in the model when no forcing is present. Finding the characteristic equation and classifying behavior based on its roots, the three damping regimes are

1. Underdamped $c < \omega_0$: In this case, the eigenvalues are distinct complex numbers and the general solution is

$$y(t) = e^{-ct} [C_1 \cos(\eta t) + C_2 \sin(\eta t)] \quad \text{where} \quad \eta = \sqrt{\omega_0^2 - c^2}. \quad (17)$$

2. Overdamped $c > \omega_0$: In this case, the eigenvalues are distinct real numbers $\lambda_1 < \lambda_2 < 0$ and the general solution is

$$y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}. \quad (18)$$

3. Critically damped $c = \omega_0$: In this case, the eigenvalues are both the same value $\lambda_1 = \lambda_2 = -c$ and the general solution is

$$y(t) = C_1 e^{-ct} + C_2 t e^{-ct}. \quad (19)$$

In all three cases the homogeneous solution $y_h(t) \rightarrow 0$ as $t \rightarrow \infty$. The underdamped case is the only solution with oscillatory damped behavior, and the critically damped case decays to equilibrium most quickly of the three cases.

Driving the oscillator with the limit cycle velocity $F(t) = \beta v_{LC}(t)$ with frequency ω_{LC} exhibits three distinct behaviors depending on whether the driving frequency is greater than, equal to, or less than the natural frequency, ω_0 . These cases, as presented in [7], are

1. Below resonance $\omega_{LC} < \omega_0$: After transient behavior decays the displacement settles into steady state motion with frequency ω_{LC} in phase with the forcing frequency ω_{LC} .
2. At resonance $\omega_{LC} = \omega_0$: After transient behavior decays the displacement settles into steady state motion 90° out of phase with the forcing frequency ω_{LC} . The amplitude grows to a maximum since the oscillator is being driven at resonant frequency.
3. Above resonance $\omega_{LC} > \omega_0$: After transient behavior decays the displacement settles into steady state motion 180° out of phase with the forcing frequency ω_{LC} . The amplitude above resonance is *less than* the amplitude when driven below resonance.

Finding the full solution to the inhomogeneous equation (16) is done by first finding the homogeneous solution $y_h(t)$, described earlier in this subsection, then finding the particular solution $y_p(t)$, and taking a linear combination arriving at the full solution, $y(t) = y_h(t) + y_p(t)$.

3 Mathematical Model

The Coupled System

Combining the theory developed in §2, the effects of coupling the Stuart-Landau equation to the damped driven harmonic oscillator equation can be studied. In this model, the transverse velocity described by the Stuart-Landau equation is used to drive the oscillator, which in turn provides feedback to the velocity in the wake. There are two different ways in which the feedback can occur,

1. The velocity of the cylinder dy/dt can be used to drive the Stuart-Landau equation on the right-hand side, as follows

$$\begin{aligned} \frac{dv}{dt} - (a_R + \imath a_I)v + (\ell_R + \imath \ell_I)|v|^2v &= \alpha \frac{dy}{dt}, \\ \frac{d^2y}{dt^2} + 2c \frac{dy}{dt} + \omega_0 y &= \beta v, \end{aligned} \tag{20}$$

where $[\alpha] = 1/s$ and $[\beta] = kg/s$ are the coupling coefficients. This coupling provides an interesting and complicated interaction between two oscillators in which the velocity in the wake is affected by the velocity of the driven cylinder.

2. The displacement of the cylinder can be coupled to the bifurcation parameter a_R as follows

$$\begin{aligned} \frac{dv}{dt} - ((a_R + \gamma|y|) + \imath a_I)v + (\ell_R + \imath \ell_I)|v|^2v &= 0, \\ \frac{d^2y}{dt^2} + 2c \frac{dy}{dt} + \omega_0 y &= \beta v, \end{aligned} \tag{21}$$

where $[\alpha] = 1/s$ and $[\gamma] = 1/m$ are the coupling coefficients. This coupling is interesting because even in the subcritical regime $Re < Re_c$ we could expect that the displacement could provide sufficient energy to the system such that the altered bifurcation parameter $a_R + \gamma|v|$ goes from subcritical (negative) to supercritical (positive).

In order to simplify the nondimensionalization, I will include *both* couplings to the Stuart-Landau equation, and let our overall coupled system be given as

$$\begin{aligned} \frac{dv}{dt} - ((a_R + \gamma|y|) + \imath a_I)v + (\ell_R + \imath \ell_I)|v|^2v &= \alpha \frac{dy}{dt}, \\ \frac{d^2y}{dt^2} + 2c \frac{dy}{dt} + \omega_0 y &= \beta v. \end{aligned} \quad (22)$$

In this way, the first coupling described by (20) corresponds to $\gamma = 0$, and the second coupling described by (21) corresponds to $\alpha = 0$

Nondimensionalization

The system (22) can be nondimensionalized using the following rescaling of variables, as done by Le Gal in [6]

$$\tau = |a_R|t, \quad (\tilde{v}, \tilde{y}) = \left(\frac{\ell_R}{|a_R|} \right)^{1/2} \left(\frac{v}{U}, \frac{y}{D} \right), \quad (23)$$

where the characteristic velocity U is taken to be the velocity of incoming flow (normal to the transverse velocity), and D is the diameter of the cylinder. It is desirable to scale both equations with the same characteristic time scale, and rescaling yields the following dimensionless system

$$\begin{aligned} \frac{d\tilde{v}}{d\tau} - ((S + \gamma'|\tilde{y}|) + \imath A)\tilde{v} + (1 + \imath L)|\tilde{v}|^2\tilde{v} &= \alpha' \frac{d\tilde{y}}{d\tau}, \\ \frac{d^2\tilde{y}}{d\tau^2} + 2C \frac{d\tilde{y}}{d\tau} + \Omega_0^2 \tilde{y} &= \beta' \tilde{v}, \end{aligned} \quad (24)$$

where the nondimensional parameters are defined as

$$\begin{aligned} S = \text{sgn}(a_R), \quad A = \frac{a_I}{|a_R|}, \quad L = \frac{\ell_I}{\ell_R}, \quad C = \frac{c}{|a_R|}, \quad \Omega_0^2 = \frac{\omega_0^2}{|a_R|^2}, \\ \alpha' = \alpha \frac{D}{U}, \quad \beta' = \frac{\beta U}{mD|a_R|^2}, \quad \gamma' = \frac{\gamma D}{(|a_R|\ell_R)^{1/2}}. \end{aligned} \quad (25)$$

Recycling variables, letting $v(t) = \rho(t)e^{i\phi(t)}$, and letting $x = \dot{y}$ to reduce the second order oscillator equation into a system of two first order equations gives the following system of four coupled nonlinear equations

$$\begin{aligned} \dot{\rho} &= (S + \gamma|y|)\rho - \rho^3 + \alpha x \cos(\phi), \\ \dot{\phi} &= A - L\rho^2 - \alpha \frac{x}{\rho} \sin(\phi), \\ \dot{x} &= -2Cx - \Omega_0^2 y + \beta \rho \cos(\phi), \\ \dot{y} &= x. \end{aligned} \quad (26)$$

In this nondimensional setting, the amplitude of the transverse velocity limit cycle is $|\tilde{v}| = 1$, and the angular frequency is $\Omega = A - L$. The natural frequency of the oscillator is Ω_0 , and critical damping occurs when $C = \Omega_0$. The system (26) is integrated and analyzed using a fourth order Runge-Kutta method.

4 Results

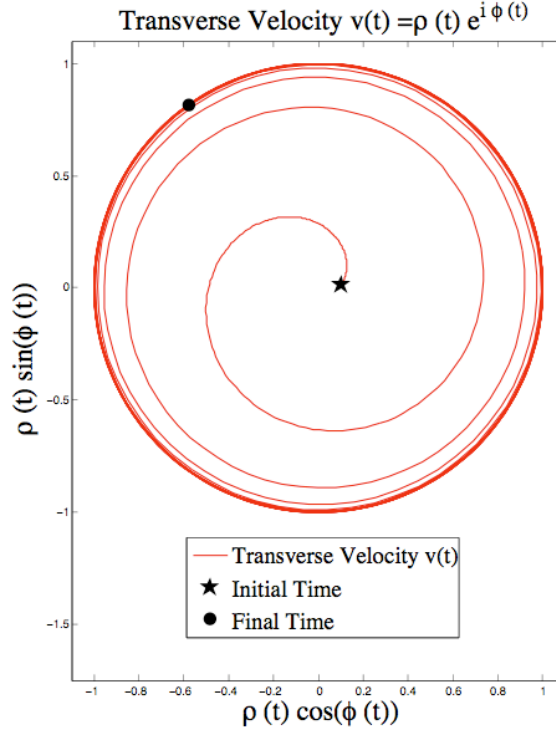


Figure 4: Phase plane of complex transverse velocity $v(t)$. The initial perturbation v_0 reaches a periodic limit cycle with amplitude $|v| = 1$ and angular velocity $\Omega = A - L$.

First it will be important to justify that the Stuart-Landau model does in fact reach a stable limit cycle in the supercritical regime. Figure 4 shows the development of a small disturbance in the velocity reaching a periodic limit cycle in the supercritical regime of vortex shedding. As expected, the steady state solution will provide a periodic driving force for the oscillator.

Now we can examine three cases of coupling between the equations:

1. $\alpha = \gamma = 0$: No feedback from the displaced cylinder to the velocity of the wake.
2. $\gamma = 0$: The velocity in the wake is driven by the velocity of the cylinder.
3. $\alpha = 0$: The bifurcation parameter is dependent on the displacement of the cylinder.

Case 1: $\alpha = \gamma = 0$

In this case, there is no feedback from the cylinder to the velocity in the wake, so we would expect to see the resonant behavior of the oscillator equation when driven by the periodic forcing. This indeed is the case, as can be shown in Figure 5. All three regimes of resonance are observed, which validates that the Stuart-Landau model provides a sufficient periodic driving force to the oscillator.

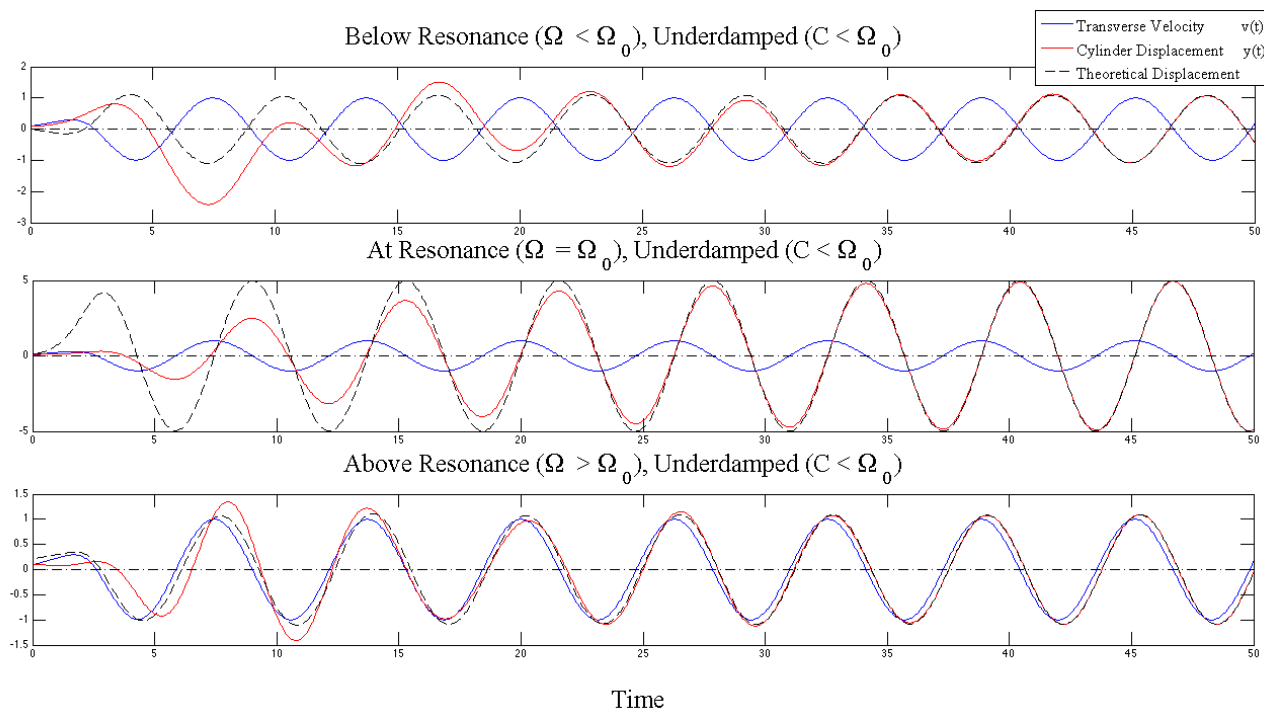


Figure 5: The different behaviors of the system, consistent with the expected results of the damped driven harmonic oscillator.

Case 2: $\gamma = 0$

In this case, the velocity in the wake is driven by the velocity of the cylinder, so we'd expect an interesting and complicated interaction between two oscillators. In the underdamped case when driven at resonance, we'd expect the cylinder to increase in amplitude and settle into a steady state 90° out of phase with the forcing. This same behavior can be observed in Figure 6 to happen when feedback is present. The velocity of the wake exhibits interesting phase plane behavior, having an "eyeball" shape which has a maximum phase plane amplitude twice that of the circular limit cycle encountered when no feedback is present. Also note that the amplitude of the cylinder displacement is twice that found previously. This makes sense physically since we expect both systems to acquire more energy from each other as time goes on.

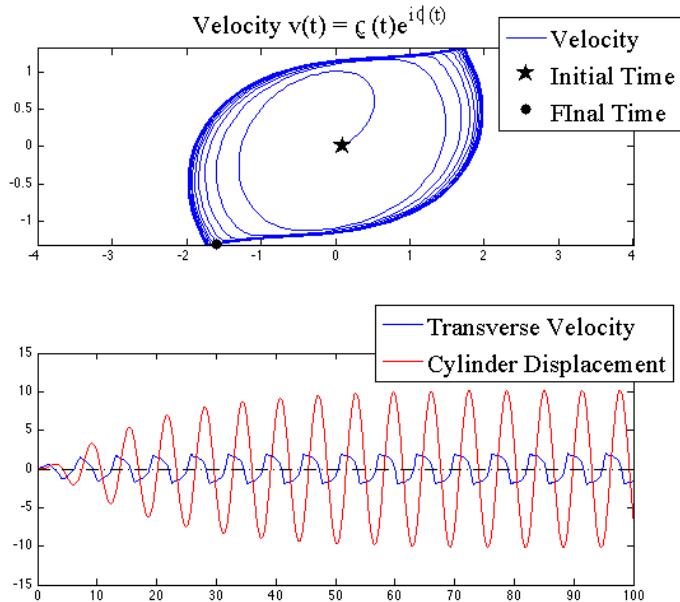


Figure 6: Velocity in the wake driven by the velocity of the cylinder. Underdamped and driven at resonant frequency.

Case 3: $\alpha = 0$

In this case, we set $Re < Re_c$, so we expect the disturbances to decay to zero, but we allow the bifurcation parameter to be dependent on the displacement of the cylinder. Thus, even in the subcritical regime, if the cylinder is initially displaced such that the quantity $a_R + \gamma|y|$ is close to the critical value of zero, there is a chance that the system might develop into periodic vortex shedding if $a_R + \gamma|y|$ goes from negative to positive. This is shown in Figure 7. The cylinder is initially displaced at a value such that $a_R + \gamma|y| > 0$, and driven at resonant frequency. The damping is then *decreased* through a critical value c_{crit} until the system exhibits periodic oscillation in the subcritical regime.

5 Conclusion and Future Work

There is a substantial amount of future work that can be done with the coupled system of equations, some of which include

1. Update the "embedding method for bluff body flows" presented by Ravoux, Nadim, and Haj-Hariri in [2] to allow the cylinder to be nonstationary and be attached to a spring. This would allow the numerical verification of the results encountered in the analysis of the coupled equations.
2. Analyzing the numerically observed closed orbit found when the cylinder velocity provides feedback to the velocity in the wake (i.e. when $\gamma = 0$ in (22)). This closed orbit could

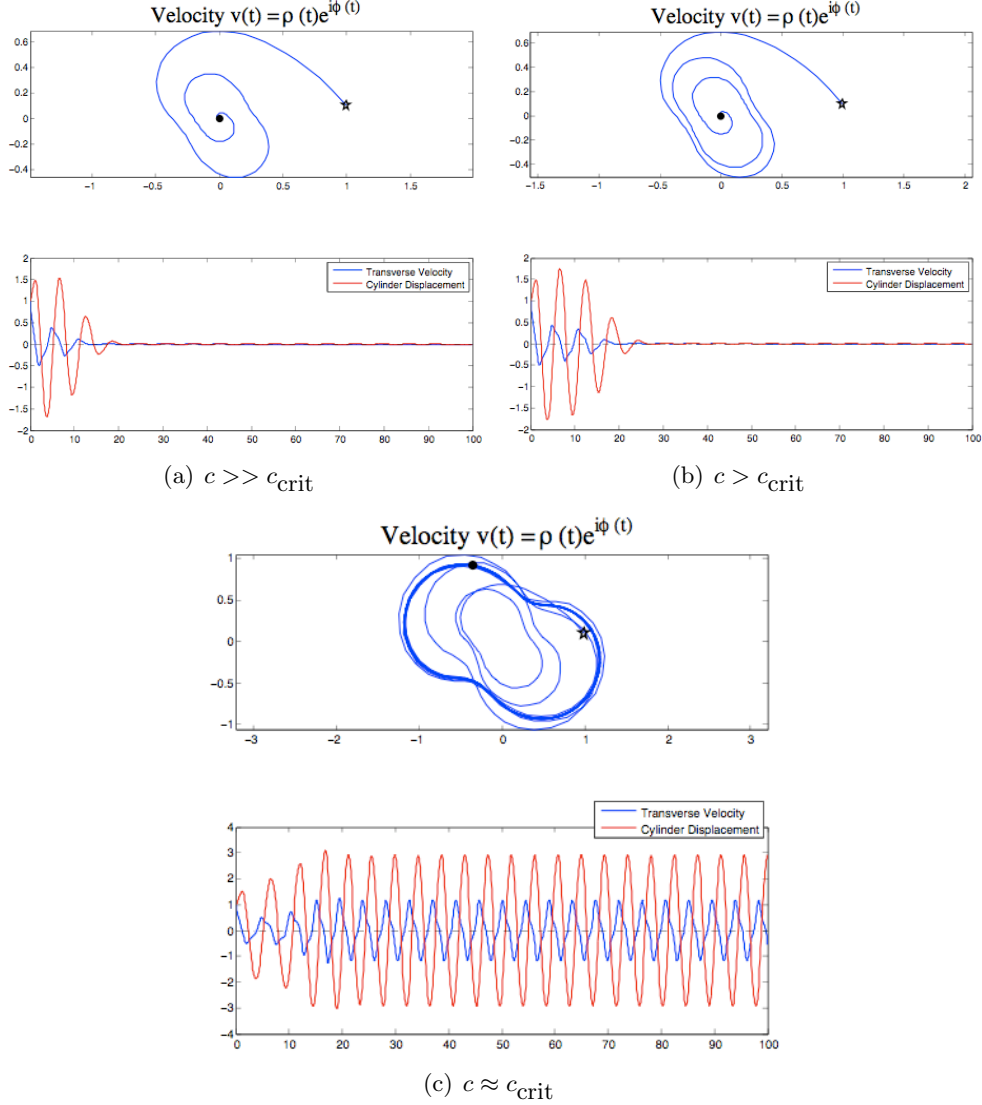


Figure 7: Velocity in the wake developing into periodic vortex shedding in the subcritical $Re < Re_c$ regime by decreasing the damping and driving at resonant frequency.

potentially be verified using the Poincaré-Bendixson theorem or ruled out by constructing a Liapunov function as described by Strogatz in [8].

3. Find the critical value of damping c_{crit} which allows for periodic oscillations in the subcritical regime $Re < Re_c$ when the bifurcation parameter is dependent on the cylinder displacement (i.e. when $\alpha = 0$ in (22)).
4. Explore the couplings described in this paper further, and explore other couplings between the two equations.

In conclusion, the Stuart-Landau model provides a description of the flow past a cylinder as Reynolds number increases through a critical value Re_c . Below this value in the subcritical regime, steady laminar flow is expected, while above this value in the postcritical regime, periodic laminar vortex shedding is expected. Both of these expectations are verified by the Stuart-Landau model, and an explicit form of the transverse velocity can be found. This velocity can then be used to drive a damped harmonic oscillator, which provides several interesting results when the two equations are coupled in different ways.

References

- [1] P. Fraunie J. Dusek, P. Le Gal. A numerical and theoretical study of the first hopf bifurcation in a cylinder wake. *Journal of Fluids and Structures*, 264:59–80, 1994.

This paper justifies using the transverse velocity to drive the cylinder on page 69.

- [2] H. Haj Hariri J. Ravoux, A. Nadim. An embedding method for bluff body flows: Interactions of two side-by-side cylinder wakes. *Theoret. Comput. Fluid Dynamics*, 16:433–466, 2003.

Numerical method that I am currently examining for my PhD research. Used to solve for flow past a cylinder. I will hopefully be able to use this embedding method to verify results found by analysis of the Stuart-Landau model.

- [3] L. Boyer M. Provansal, C. Mathis. Benard-von karman instabilities: Transient and forced regimes. *Journal of Fluid Mechanics*, 182:1–22, 1987.

Good discussion of Landau model, physical derivation and experimental observations.

- [4] P. Le Gal M. Thompson. Application of the forced stuart-landau model to cylinder wake oscillation. *14th Australasian Fluid Mechanics Conference, Adelaide University, Adelaide, Australia, Dec. 10-14*, pages 1–22, 2001.

Brief discussion of the forced Stuart-Landau model with some theory and results.

- [5] P. Le Gal M. Thompson. The stuart-landau model applied to wake transition revisited. *European Journal of Mechanics B/Fluids*, 23:219–228, 2004.

Discusses the un-forced Stuart-Landau equation with plenty of results and figures to reinforce the basics.

- [6] M. Thompson P. Le Gal, A. Nadim. Hysteresis in the forced stuart-landau equation: Application to vortex shedding from an oscillating cylinder. *Journal of Fluids and Structures*, 15:1–13, 2001.

This paper presents the forced Stuart-Landau equation, which will be the basis of my research project.

- [7] J. Polking. *Differential Equations with Boundary Value Problems*. Pearson Prentice Hall, 12006.

Textbook for DE's describing DDHO.

- [8] Steven H. Strogatz. *Nonlinear Dynamics and Chaos*. Perseus Books Publishing, LLC, 1994.

Textbook for our course, gives good background on Hopf bifurcations.

- [9] C. Williamson. Vortex dynamics in the cylinder wake. *Annu. Rev. Fluid Mech.*, 28:477–539, 1996.

This is a very broad overview of the dynamics of fluid flow past a cylinder, so it will give me a good foundation of knowledge about the subject.