This article appeared in Forum Mathematicum **16** (2004), no. 1, 97-107.

RADON TRANSFORMS AND THE FINITE GENERAL LINEAR GROUPS

MICHAEL E. ORRISON

Abstract. Using a class sum and a collection of related Radon transforms, we present a proof G. James's Kernel Intersection Theorem for the complex unipotent representations of the finite general linear groups. The approach is analogous to that used by F. Scarabotti for a proof of James's Kernel Intersection Theorem for the symmetric group. In the process, we also show that a single class sum may be used to distinguish between distinct irreducible unipotent representations.

1. Introduction

The work of G. James [6] reveals interesting similarities between the representations of the symmetric group and the unipotent representations of the finite general linear groups. One such similarity is that both enjoy a Kernel Intersection Theorem which characterizes their irreducible modules as the intersections of kernels of certain operators. Using the class sum of transpositions and a collection of related Radon transforms, F. Scarabotti [8] has given a short proof of this characterization for the complex representations of the symmetric group. In the spirit of strengthening the relationship between the symmetric group and the finite general linear groups, we present an analogous approach to the Kernel Intersection Theorem for the complex unipotent representations of the finite general linear groups. In doing so, we also show that a single class sum may be used to distinguish between distinct irreducible unipotent representations.

2. Background

Our approach requires a few facts from the representation theory of finite groups, a sense of how to create operators from incidence relations, and a familiarity with compositions, partitions, and Gaussian polynomials.

Representation Theory. We begin by recalling a few facts from the representation theory of finite groups. A good reference is [9].

Let G be a finite group and let $\mathbb{C}[G]$ be the complex group algebra of G. Recall that a (*complex*) *representation* of G is a $\mathbb{C}[G]$-module M, and that if $C_1, \ldots, C_h$ are

Date: December 18, 2001.

1991 *Mathematics Subject Classification.* 20C33, 43A85, 44A12.

Key words and phrases. Radon transform, finite general linear group, Kernel Intersection Theorem.

the distinct conjugacy classes of G, then there are h distinct (up to isomorphism) irreducible $\mathbb{C}[G]$-modules, say $W_1, \ldots, W_h$. Let χ_j be the character of W_j and $\chi_j(C_i)$ be the value of χ_j on C_i.

Every $\mathbb{C}[G]$-module M is semisimple and may therefore be written as a direct sum of irreducible submodules, say $U_1, \ldots, U_l$. Denote by M_i the direct sum of those $U_1, \ldots, U_l$ that are isomorphic to W_i. This creates the *isotypic decomposition*

$$M = M_1 \oplus \cdots \oplus M_n$$

of M where M_i is then the W_i-*isotypic subspace* of M.

Let C be a conjugacy class of G and let T be the class sum of C in $\mathbb{C}[G]$, that is,

$$T = \sum_{c \in C} c.$$

If U is an irreducible $\mathbb{C}[G]$-module with character χ, then U is an eigenspace of T with eigenvalue $|C|\chi(C)/\dim U$. Thus, if T_i is the class sum of the conjugacy class C_i, then the W_j-isotypic subspace M_j of M is an eigenspace of T_i with eigenvalue $|C_i|\chi_j(C_i)/\dim U_j$.

Radon Transforms. Let G act on two finite sets X and Y, and let M and N be the associated $\mathbb{C}[G]$-permutation modules, respectively. Furthermore, suppose there is an incidence relation between X and Y that is invariant under the action of G. We write $x \sim y$ if $x \in X$ is incident to $y \in Y$, and we define the *Radon transform* $R : M \to N$ by

$$R(x) = \sum_{y : x \sim y} y$$

(see [1]). Because the incidence relation is invariant under the action of G, the Radon transform R is a $\mathbb{C}[G]$-module homomorphism. The adjoint $R^* : N \to M$ is defined by

$$R^*(y) = \sum_{x : x \sim y} x.$$

The map R^*R is therefore a $\mathbb{C}[G]$-module homomorphism from M to M.

Compositions and Partitions. If n is a positive integer, then a *composition* of n is a sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of non-negative integers whose sum is n. If $\lambda_1 \geq \cdots \geq \lambda_m > 0$, then λ is a *partition* of n. To each composition λ, there is a corresponding partition $\overline{\lambda}$ obtained by arranging the positive parts of λ in non-increasing order. For example, if $\lambda = (1, 3, 0, 3, 2, 0)$, then $\overline{\lambda} = (3, 3, 2, 1)$.

The partitions of n form a partially ordered set under the *dominance order* where, given two partitions μ and λ, μ *dominates* λ if

$$\mu_1 + \cdots + \mu_i \geq \lambda_1 + \cdots + \lambda_i$$

for all $i \geq 1$. If μ dominates λ, we write $\mu \trianglerighteq \lambda$. If μ dominates λ and $\mu \neq \lambda$, then we write $\mu \triangleright \lambda$.

Gaussian Polynomials. Let $\mathbb{F}_q$ be the finite field with q elements. If k is a non-negative integer, define $[k] = 1 + q + q^2 + \cdots + q^{k-1}$. Define $[k]! = [k][k-1]\cdots[1]$ if $k > 0$, and $[0]! = 1$. Next, define

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \frac{[n]!}{[k]![n-k]!} & \text{if } n \geq k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This is a polynomial in q, a *Gaussian polynomial*, and is equal to the binomial coefficient $\binom{n}{k}$ when $q = 1$ (see, e.g., [10]).

We will make use of the following theorem and its corollaries. Proofs may be found in [6].

Theorem 1. *Let V_1 and V_2 be subspaces of an n-dimensional vector space V over $\mathbb{F}_q$. Let $\dim V_1 = d_1$ and $\dim V_2 = d_2$. If $V_1 \cap V_2 = \langle 0 \rangle$, then the number of k-dimensional subspaces W of V such that $W \cap V_1 = \langle 0 \rangle$ and $W \supseteq V_2$ is*

$$q^{d_1(k-d_2)} \begin{bmatrix} n - d_1 - d_2 \\ k - d_2 \end{bmatrix}.$$

Corollary 2. *If $V_1 \supseteq V_2$, then the number of k-dimensional subspaces W of V such that $W \cap V_1 = V_2$ is*

$$q^{(d_1-d_2)(k-d_2)} \begin{bmatrix} n - d_1 \\ k - d_2 \end{bmatrix}.$$

Corollary 3. *The number of k-dimensional subspaces W of V such that $W \supseteq V_1$ is*

$$\begin{bmatrix} n - d_1 \\ k - d_1 \end{bmatrix}.$$

3. The Finite General Linear Groups

Let $\mathbb{F}_q$ be the field of q elements, let $n > 1$, and let V be an n-dimensional vector space over $\mathbb{F}_q$ with basis $e_1, \ldots, e_n$. The general linear group $GL_n(q) = G_n$ is, by definition, the group of automorphisms of V. For convenience, we will identify G_n with the group of non-singular $n \times n$ matrices over $\mathbb{F}_q$ where the automorphism given by the matrix (g_{ij}) is the one for which $e_j \mapsto \sum_{i=1}^{n} g_{ij} e_i$.

Note. *We assume that $q \neq 2$ and treat the case $q = 2$ separately.*

Let $V_0, \ldots, V_m$ be a collection of subspaces of V such that $V_0 = V$, $V_m = \langle 0 \rangle$, and

$$V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{m-1} \supseteq V_m.$$

Let d_i be the dimension of V_i. We say that $V_0, \ldots, V_m$ form a *flag of type* $\lambda = (\lambda_1, \ldots, \lambda_m)$ where $\lambda_i = d_{i-1} - d_i$. We denote by X_λ the set of all such flags of type λ. Note that λ is a composition of n and that, for any composition μ of n, we can always find a flag of type μ.

The action of G_n on V induces a transitive action of G_n on X_λ. The resulting $\mathbb{C}[G_n]$-permutation module M_λ is called a *unipotent representation* of G_n. Although not obvious, M_λ is isomorphic to $M_{\overline{\lambda}}$ (Theorem 14.7 in [6]), so we will assume λ is a partition of n.

The collection of irreducible submodules (up to isomorphism) of unipotent representations of G_n are indexed by the partitions of n (Corollary 16.4 in [6]). If μ is a partition of n, then we denote the corresponding irreducible $\mathbb{C}[G_n]$-module by S_μ. By Theorem 15.16 in [6],

$$M_\lambda \cong \bigoplus_{\mu \trianglerighteq \lambda} \kappa_{\mu\lambda} S_\mu$$

where the $\kappa_{\mu\lambda}$ are the usual *Kostka numbers* (see, e.g., [7]) and denote the multiplicity of S_μ in M_λ. In particular, $\kappa_{\lambda\lambda} = 1$, so we may unambiguously identify S_λ with the irreducible submodule of M_λ to which it is isomorphic.

The following lemma is Corollary 11.14 (iii) in [6]:

Lemma 4. *$S_\lambda \subseteq \bigcap_\theta \ker\theta$, the intersection being over all $\mathbb{C}[G_n]$-homomorphisms θ which map M_λ into some M_μ with $\mu \rhd \lambda$.*

The Kernel Intersection Theorem for G_n (Theorem 15.19 in [6]) states that the above inclusion is actually equality. In what follows, we use a class sum and a collection of related Radon transforms to show this. Our approach is analogous to that used by F. Scarabotti in [8] for a proof of G. James's Kernel Intersection Theorem for the symmetric group S_n.

4. Reflections and Radon Transforms

In this section, we show how the conjugacy class of reflections in G_n is related to a collection of Radon transforms. We begin by proving several useful facts about reflections.

Reflections. Recall that V is an n-dimensional vector space over $\mathbb{F}_q$ with basis $e_1, \ldots, e_n$. Let C be the conjugacy class of G_n that contains the automorphism of V that transposes e_1 and e_2 while fixing the other basis vectors. The matrix corresponding to this automorphism is

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

Each element of C has order 2 and fixes a *hyperplane* (a codimension-1 subspace of V) pointwise. We will refer to C as the conjugacy class of *reflections* in G_n.

Fix a flag $x = V_0 \supset \cdots \supset V_m$ where $d_i = \dim V_i$. The flag x determines a useful partition $\{P_1(x), \ldots, P_m(x)\}$ of the set of hyperplanes of V where we say that a hyperplane H is in $P_j(x)$ if H contains V_j but not V_{j-1}.

Lemma 5. *$P_j(x)$ contains $q^{n-d_{j-1}}[d_{j-1} - d_j]$ hyperplanes.*

Proof. By Corollary 3, there are

$$\begin{bmatrix} d_{j-1} - d_j \\ (d_{j-1} - 1) - d_j \end{bmatrix} = [d_{j-1} - d_j]$$

codimension-1 subspaces of V_{j-1} that contain V_j. If $H \in P_j(x)$, then $H \cap V_{j-1}$ is one such subspace. Thus, by Corollary 2, to each such subspace there correspond

$$q^{(d_{j-1}-(d_{j-1}-1))((n-1)-(d_{j-1}-1))} \begin{bmatrix} n - d_{j-1} \\ (n-1) - (d_{j-1} - 1) \end{bmatrix} = q^{n-d_{j-1}}$$

hyperplanes. □

Let $c \in C$ and suppose that c fixes the hyperplane $H \in P_j(x)$ pointwise. Fix $v \in V_{j-1} - (H \cap V_{j-1})$. The vectors v and cv are transposed by c since c^2 is the identity. It follows that $v + cv \in H$, thus $cv = -v + h$ for some $h \in H$. Let i be such that cv is contained in V_{i-1} but not V_i. Note that $i \leq j$ and that each vector in $V_{j-1} - (H \cap V_{j-1})$ gives rise to the same i. Next, define $\varphi(x, c) = (i, j)$. It is easy to show that $x = cx$ if and only if $i = j$, and that if $i < j$, then

$x \cap cx = (V_0 \cap cV_0) \supseteq \cdots \supseteq (V_m \cap cV_m)$ is a flag of type $\mu = (\mu_1, \ldots, \mu_m)$ where $\mu_i = \lambda_i + 1$, $\mu_j = \lambda_j - 1$, and $\mu_k = \lambda_k$ for $k \neq i, j$.

Lemma 6. *Let c be a reflection, let x be the flag $V_0 \supset \cdots \supset V_m$, and let $d_i = \dim V_i$. If $1 \leq i < j \leq m$ and $\varphi(x, c) = (i, j)$, then there are $q^{n-2+d_{j-1}-d_i}(q-1)$ reflections that map x to cx.*

Proof. Fix a vector $v \in V_{j-1} - (V_{j-1} \cap cV_{j-1})$. To map V_{j-1} to cV_{j-1} using a reflection, we may first send v to any one of the $q^{d_{j-1}} - q^{d_{j-1}-1}$ vectors $v' \in cV_{j-1} - (V_{j-1} \cap cV_{j-1})$. The hyperplane corresponding to our reflection must then contain the subspace $\langle V_i \cap cV_i, v + v' \rangle$ of dimension d_i, but not the subspace $\langle v \rangle$. By Theorem 1, there are

$$q^{(1)((n-1)-d_i)} \begin{bmatrix} n-1-d_i \\ (n-1)-d_i \end{bmatrix} = q^{(n-1)-d_i}$$

such hyperplanes. It follows that there are

$$(q^{d_{j-1}} - q^{d_{j-1}-1})q^{(n-1)-d_i} = q^{n-2+d_{j-1}-d_i}(q-1)$$

reflections that map x to cx. □

Proposition 7. *Let c be a reflection, let x be the flag $V_0 \supset \cdots \supset V_m$, and let $d_i = \dim V_i$. If $1 \leq i < j \leq m$, then there are $q^{2(d_i - d_{j-1})+1}[d_{i-1} - d_i][d_{j-1} - d_j]$ flags y such that $y = cx$ for some reflection c where $\varphi(x, c) = (i, j)$.*

Proof. Let $H \in P_j(x)$ and let $v \in V_{j-1} - (H \cap V_{j-1})$. Since $|H \cap (V_{i-1} - V_i)| = q^{d_{i-1}-1} - q^{d_i-1}$, there are $q^{d_{i-1}-1} - q^{d_i-1}$ reflections c such that $\varphi(x, c) = (i, j)$ where c fixes H pointwise. It follows that there are

$$|P_j(x)|(q^{d_{i-1}-1} - q^{d_i-1}) = q^{n+d_i-d_{j-1}-1}(q^{d_{i-1}-d_i} - 1)[d_{j-1} - d_j]$$

reflections c such that $\varphi(x, c) = (i, j)$.

By Lemma 6, if $\varphi(x, c) = (i, j)$ and $y = cx$, then there are $q^{n-2+d_{j-1}-d_i}(q-1)$ reflections that map x to y. It follows that there are

$$\frac{q^{n+d_i-d_{j-1}-1}(q^{d_{i-1}-d_i} - 1)[d_{j-1} - d_j]}{q^{n-2+d_{j-1}-d_i}(q-1)} = q^{2(d_i-d_{j-1})+1}[d_{i-1} - d_i][d_{j-1} - d_j]$$

flags y such that $y = cx$ for some reflection c where $\varphi(x, c) = (i, j)$. □

Lemma 8. *If x is the flag $V_0 \supset \cdots \supset V_m$ and $d_i = \dim V_i$, then there are $\sum_{j=1}^{m} q^{n-1}[d_{j-1} - d_j]$ reflections that fix x.*

Proof. If $c \in C$ fixes x, then $\varphi(x, c) = (j, j)$ for some j where c fixes a hyperplane $H \in P_j(x)$ pointwise. Therefore, for each hyperplane $H \in P_j(x)$, choose some $v \in V_{j-1} - (H \cap V_{j-1})$. If $c \in C$ is to fix x while fixing the hyperplane H pointwise, then c must map v to some v' such that $v + v' \in H \cap V_{j-1}$. As there are $q^{d_{j-1}-1}$ such v', it follows that there are

$$\sum_{j=1}^{m} |P_j(x)| q^{d_{j-1}-1} = \sum_{j=1}^{m} q^{n-d_{j-1}}[d_{j-1} - d_j] q^{d_{j-1}-1} = \sum_{j=1}^{m} q^{n-1}[d_{j-1} - d_j]$$

reflections that fix the chain x. □

Radon Transforms. Using the results above, we now relate the conjugacy class of reflections to a collection of Radon transforms. We begin by considering the class sum of reflections in $\mathbb{C}[G_n]$.

Let x be the flag $V_0 \supset \cdots \supset V_m$ of type $\lambda = (\lambda_1, \ldots, \lambda_m)$ (recall that we are assuming λ is a partition of n). Let $1 \leq i < j \leq m$ and let $\mu = (\mu_1, \ldots, \mu_m)$ where $\mu_i = \lambda_i + 1$, $\mu_j = \lambda_j - 1$, and $\mu_k = \lambda_k$ for $k \neq i, j$. We say that x is *ij-incident* to $y \in X_\mu$ if y is a flag of the form

$$V_0 \supset \cdots \supset V_{i-1} \supset H \cap V_i \supset \cdots \supset H \cap V_{j-1} \supseteq V_j \supset \cdots \supset V_m$$

for some hyperplane $H \in P_j(x)$. This incidence relation is invariant under the action of G_n, thus the associated Radon transform $R_{ij} : M_\lambda \to M_\mu$ is a $\mathbb{C}[G_n]$-module homomorphism. Moreover, note that $\mu \rhd \lambda$. We therefore have

Lemma 9. *$\bigcap_\theta \ker \theta \subseteq \bigcap_{1 \leq i < j \leq m} \ker R_{ij}$ where θ ranges over all $\mathbb{C}[G_n]$-module homomorphisms that map M_λ into some M_μ with $\mu \rhd \lambda$.*

Let T be the class sum of reflections, let $1 \leq i < j \leq m$, and define $T_{ij} : M_\lambda \to M_\lambda$ by $T_{ij}(x) = \sum x'$, where the sum is over all x' such that $x' = cx$ for some $c \in C$ and $\varphi(x, c) = (i, j)$. Let $I : M_\lambda \to M_\lambda$ be the identity map. By Lemma 6 and Lemma 8, we have

Lemma 10. *$T = \sum_{1 \leq i < j \leq m} q^{n-2+d_{j-1}-d_i}(q-1)T_{ij} + \sum_{j=1}^{m} q^{n-1}[d_{j-1} - d_j]I.$*

Define $T_{jj} = R_{jj} = I$ for $1 \leq j \leq m$. The T_{ij} and R_{ij} are related:

Lemma 11. *If $1 \leq i < j \leq m$, then $R_{ij}^* R_{ij} = \sum_{i \leq k \leq j} \alpha_{ij}^k T_{kj}$ where*

$$\alpha_{ij}^k = \begin{cases} 1 & \text{if } k = i, \\ q^{d_i - d_k - 1}(q-1) & \text{if } i < k < j, \\ q^{d_i - d_{j-1}}[d_{j-1} - d_j] & \text{if } k = j. \end{cases}$$

Proof. Let $x, x' \in X_\lambda$ where x is the flag $V_0 \supset \cdots \supset V_m$. If x and x' are ij-incident to the same $y \in X_\mu$, then it is easy to show that there is a $c \in C$ such that $cx = x'$ and $\varphi(x, c) = (k, j)$, where $i \leq k \leq j$. We simply need to count the number of such y for each such pair x and x'. If $k = j$, then $x = x'$. Thus α_{ij}^j is the number of flags to which x is ij-incident, which is the number of codimension-1 subspaces of V_i that contain V_j but not V_{j-1}, or $q^{d_i - d_{j-1}}[d_{j-1} - d_j]$.

If $k \neq j$, then $x \neq x'$, and the number of flags y which are ij-incident to both x and x' is the number of codimension-1 subspaces of V_i that contain $V_k \cap cV_k$ but not V_{j-1} or cV_{j-1}. To compute this number, we use the Principle of Inclusion and Exclusion and the fact that $\dim(V_k \cap cV_k) = d_k - 1$, $\dim\langle V_k \cap cV_k, V_{j-1}\rangle = \dim\langle V_k \cap cV_k, cV_{j-1}\rangle = d_k$, and $\dim\langle V_k \cap cV_k, V_{j-1}, cV_{j-1}\rangle = d_k + 1$. Thus, for $i \leq k < j$,

$$\alpha_{ij}^k = \begin{bmatrix} d_i - (d_k - 1) \\ (d_i - 1) - (d_k - 1) \end{bmatrix} - 2\begin{bmatrix} d_i - d_k \\ (d_i - 1) - d_k \end{bmatrix} + \begin{bmatrix} d_i - (d_k + 1) \\ (d_i - 1) - (d_k + 1) \end{bmatrix}$$

which is $q^{d_i - d_k - 1}(q-1)$ if $i < k$, and 1 if $i = k$. □

We may now express the T_{ij} in terms of the $R_{ij}^* R_{ij}$:

Lemma 12. *If* $1 \le i < j \le m$, *then* $T_{ij} = \sum_{i \le k \le j} \beta_{ij}^k R_{kj}^* R_{kj}$ *where*

$$\beta_{ij}^k = \begin{cases} 1 & \text{if } k = i, \\ -q^{d_i - d_k - (k-i)}(q-1) & \text{if } i < k < j, \\ -q^{d_i - d_{j-1} - (j-i-1)}[d_{j-1} - d_j] & \text{if } k = j. \end{cases}$$

Proof. We begin by noting that, when $i < j$, $T_{ij} = R_{ij}^* R_{ij} - \sum_{i<l\le j} \alpha_{ij}^l T_{lj}$. This shows that $\beta_{ij}^i = 1$. If we let $\alpha_{jj}^j = \beta_{jj}^j = 1$, then we also have the recurrence relation

$$\beta_{ij}^k = -\sum_{i<l\le k} \alpha_{ij}^l \beta_{lj}^k$$

for $i < k \le j$. We proceed by induction on $j - i$. First, suppose $k = j$. Then

$$\begin{aligned} \beta_{i,i+1}^{i+1} = -\alpha_{i,i+1}^{i+1} &= -q^{d_i - d_{(i+1)-1}}[d_{(i+1)-1} - d_{i+1}] \\ &= -q^{d_i - d_{(i+1)-1} - ((i+1)-i-1)}[d_{(i+1)-1} - d_{i+1}], \end{aligned}$$

showing that our formula for β_{ij}^j holds if $(j - i) = 1$. Assume that the formula holds for β_{lj}^j for $i < l < j$. Then, by our recurrence relation, we have

$$\begin{aligned} \beta_{ij}^j &= -\alpha_{ij}^j - \sum_{i<l<j} \alpha_{ij}^l \beta_{lj}^j \\ &= -q^{d_i - d_{j-1}}[d_{j-1} - d_j] - \sum_{i<l<j} q^{d_i - d_l - 1}(q-1)(-q^{d_l - d_{j-1} - (j-l-1)}[d_{j-1} - d_j]) \\ &= [d_{j-1} - d_j](-q^{d_i - d_{j-1}} + (q-1)(q^{d_i - d_{j-1} - (j-(i+1))} + \cdots + q^{d_i - d_{j-1} - 1})) \\ &= -q^{d_i - d_{j-1} - (j-i-1)}[d_{j-1} - d_j]. \end{aligned}$$

Next, suppose $i < k < j$. We then have

$$\begin{aligned} \beta_{i,i+2}^{i+1} = -\alpha_{i,i+2}^{i+1} \beta_{i+1,i+2}^{i+1} &= -q^{d_i - d_{i+1} - 1}(q-1) \\ &= -q^{d_i - d_{i+1} - ((i+1)-i)}(q-1) \end{aligned}$$

showing that our formula holds if $j - i = 2$. Assume that the formula for β_{lj}^k holds for $i < l < k < j$. Then

$$\begin{aligned} \beta_{ij}^k &= -\alpha_{ij}^k - \sum_{i<l<k} \alpha_{ij}^l \beta_{lj}^k \\ &= -q^{d_i - d_k - 1}(q-1) - \sum_{i<l<k} q^{d_i - d_l - 1}(q-1)(-q^{d_l - d_k - (k-l)}(q-1)) \\ &= (q-1)(-q^{d_i - d_k - 1} + (q-1)(q^{d_i - d_k - (k-i)} + \cdots q^{d_i - d_k - 2})) \\ &= -q^{d_i - d_k - (k-i)}(q-1). \end{aligned}$$

□

Now that we are able to express each of the T_{ij} in terms of the $R_{ij}^* R_{ij}$, we may express the class sum T of reflections in terms of the $R_{ij}^* R_{ij}$:

Theorem 13. *Let T be the class sum of reflections in G_n and let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be a partition of n. Viewed as an operator on M_λ, T may be written as*

$$T = \sum_{1 \leq i < j \leq m} q^{n-1+d_{j-1}-d_i-i}(q-1)R_{ij}^* R_{ij} + \sum_{j=1}^{m} q^{n-j}[d_{j-1} - d_j]I$$

where the R_{ij} are defined as above.

Proof. By Lemma 10,

$$T = \sum_{1 \leq i < j \leq m} q^{n-2+d_{j-1}-d_i}(q-1)T_{ij} + \sum_{j=1}^{m} q^{n-1}[d_{j-1} - d_j]I.$$

Therefore, by Lemma 12,

$$T = \sum_{1 \leq k < j \leq m} q^{n-2+d_{j-1}-d_k}(q-1)(\sum_{k \leq i \leq j} \beta_{kj}^i R_{ij}^* R_{ij}) + \sum_{j=1}^{m} q^{n-1}[d_{j-1} - d_j]I.$$

If $i < j$, then $R_{ij}^* R_{ij}$ will occur

$$\begin{aligned} &\sum_{1 \leq k \leq i} q^{n-2+d_{j-1}-d_k}(q-1)\beta_{kj}^i \\ &= q^{n-2+d_{j-1}-d_i}(q-1) + \sum_{1 \leq k < i} -q^{n-2+d_{j-1}-d_i-(i-k)}(q-1)^2 \\ &= q^{(n-1)+d_{j-1}-d_i-i}(q-1) \end{aligned}$$

times in the sum, and the identity will occur

$$\begin{aligned} &\sum_{1 \leq i < j} q^{n-2+d_{j-1}-d_i}(q-1)\beta_{ij}^j + \sum_{j=1}^{m} q^{n-1}[d_{j-1} - d_j] \\ &= \sum_{j=1}^{m} \left(q^{n-1}[d_{j-1} - d_j] + \sum_{1 \leq i < j} -q^{n-1-(j-i)}(q-1)[d_{j-1} - d_j] \right) \\ &= \sum_{j=1}^{m} q^{n-j}[d_{j-1} - d_j] \end{aligned}$$

times in the sum. □

By Theorem 13, since $\lambda_j = d_{j-1} - d_j$ for $1 \leq j \leq m$, we immediately have

Corollary 14. $\bigcap_{1 \leq i < j \leq m} \ker R_{ij}$ *is an eigenspace of T with eigenvalue*

$$r_\lambda = \sum_{j=1}^{m} q^{n-j}[\lambda_j].$$

By Lemma 4, we also have

Corollary 15. *The irreducible unipotent representation S_λ is an eigenspace of T with eigenvalue $r_\lambda = \sum_{j=1}^{m} q^{n-j}[\lambda_j]$.*

Theorem 16. *If λ and μ are partitions of n, then $r_\lambda = r_\mu$ if and only if $\lambda = \mu$.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_k)$ be partitions of n. Thus $r_\lambda = q^{n-1}[\lambda_1] + \cdots + q^{n-m}[\lambda_m]$ and $r_\mu = q^{n-1}[\mu_1] + \cdots + q^{n-k}[\mu_k]$. Without loss of generality, assume $k \leq m$. If $k < m$, then q^{n-m} divides both r_λ and r_μ, although $r_\lambda / q^{n-m} (\mathrm{mod}\ q) = 1 \neq 0 = r_\mu / q^{n-m} (\mathrm{mod}\ q)$. Thus $r_\lambda \neq r_\mu$. If $k = m$, subtract 1 from each part of λ and μ to create two partitions λ' and μ' of $n - m$ so that $r_\lambda = q^{n-m}[m] + q^{m+1} r_{\lambda'}$ and $r_\mu = q^{n-m}[m] + q^{m+1} r_{\mu'}$. We may repeat the argument above to show that λ' must have the same number of parts as μ', otherwise $r_{\lambda'} \neq r_{\mu'}$ and, therefore, $r_\lambda \neq r_\mu$. We then continue to repeat the process noting that the number of parts at each step is equal if and only if $\lambda = \mu$. □

By Corollary 15 and Theorem 16, we may use the class sum of reflections to distinguish between distinct irreducible unipotent representations of G_n:

Theorem 17. *Let M be a unipotent representation of G_n. If T is the class sum of reflections in G_n and $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a partition of n, then the S_λ-isotypic subspace of M is the unique eigenspace of T with eigenvalue $r_\lambda = \sum_{j=1}^{m} q^{n-j}[\lambda_j]$.*

We may now state and prove the Kernel Intersection Theorem for the complex unipotent representations of the finite general linear groups.

Theorem 18. (James) *$S_\lambda = \bigcap_\theta \ker \theta$, the intersection being over all $\mathbb{C}[G_n]$-homomorphisms θ which map M_λ into some M_μ with $\mu \rhd \lambda$.*

Proof. By Lemma 4, we know that $S_\lambda \subseteq \bigcap_\theta \ker \theta$. By Lemma 9, $\bigcap_\theta \ker \theta \subseteq \bigcap_{1 \leq i < j \leq m} \ker R_{ij}$. Thus, by Corollary 14 and Theorem 17, $\bigcap_\theta \ker \theta$ is contained in the S_λ-isotypic subspace of M_λ. Thus $S_\lambda = \bigcap_\theta \ker \theta$. □

The Case $\mathbf{q = 2}$. When $q = 2$, we need only make one change to Lemma 8:

Lemma 19. *If $q = 2$, then there are $\sum_{j=1}^{m} q^{n-1}[d_{j-1} - d_j] - [n]$ reflections that fix the chain x.*

Proof. The proof is essentially the proof for Lemma 8 with the slight change that each v has $q^{d_{j-1}-1} - 1$ choices for a v' since $-v = v$ when $q = 2$. □

This small change preserves each of the previous results up to the addition of some scalar of the identity. Therefore, with a few modifications, everything goes through as before.

References

[1] Bolker, E.: The finite Radon transform. Contemporary Mathematics 63 (1987), 27-50.

[2] Diaconis, P.: Group Representations in Probability and Statistics. Inst. of Math. Stat., Hayward, CA 1988.

[3] Fulton, W.: Young Tableaux With Applications to Representation Theory and Geometry. Cambridge University Press, Cambridge 1997.

[4] James, G.: The irreducible representations of the symmetric groups. Bull. London Math. Soc. 8 (1976), 229-232.

[5] James, G.: A characteristic-free approach to the representation theory of S_n. Journal of Algebra 46 (1977), 430-450.

[6] James, G.: Representations of General Linear Groups. Cambridge University Press, Cambridge 1984.

[7] Sagan, B.: The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions 2nd ed. Springer, New York 2001.

[8] Scarabotti, F.: Radon transforms on the symmetric group and harmonic analysis of a class of invariant Laplacians. Forum Math. 10 (1998), 407-411.

[9] Serre, J.-P.: Linear Representations of Finite Groups. Springer-Verlag, New York-Heidelberg-Berlin 1977.

[10] Stanley, R.: Enumerative combinatorics. Vol. 1. Cambridge University Press, Cambridge 1997.

Department of Mathematics, Harvey Mudd College, Claremont, CA 91711

E-mail address: `orrison@hmc.edu`