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REFLECTIONS ACTING EFFICIENTLY ON A BUILDING

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Abstract. We show how Radon transforms may be used to apply efficiently the class sum of reflections in the finite general linear group $GL_n(\mathbb{F}_q)$ to vectors in permutation modules arising from the action of $GL_n(\mathbb{F}_q)$ on the building of type $A_{n-1}(\mathbb{F}_q)$.

1. Introduction

Let G be a finite group acting transitively on a finite set X, and let M be the resulting $\mathbb{C}[G]$ permutation module. If C is a conjugacy class in G, then its class sum $T = \sum_{c \in C} c \in \mathbb{C}[G]$ may be viewed as a diagonalizable linear transformation $T : M \to M$. The question addressed in this paper is the following:

Given $f \in M$, how may we compute Tf efficiently?

This question arises in *spectral analysis*, which is a non-model based approach to the analysis of data arising as a complex-valued function f on a set X with a group G of automorphisms. Developed by Diaconis in [6, 7], the subject extends the classical spectral analysis of time series and requires the projection of f onto $\mathbb{C}[G]$-invariant subspaces of M.

Class sums play an important role in spectral analysis because their eigenspaces are direct sums of fundamental invariant subspaces known as *isotypic subspaces* or *homogeneous components*. With a suitable collection of class sums, isotypic projections may be achieved as eigenspace projections, and being able to apply class sums efficiently allows us to make effective use of iterative eigenspace projection techniques [10, 11].

In this paper, we use Radon transforms to show how the class sum T of reflections in the finite general linear group may be applied surprisingly efficiently (see Theorem 6 and Theorem 8) when the underlying set is a set of residues of the building of type $A_{n-1}(\mathbb{F}_q)$. Our work builds directly upon that found in [12] in which Radon transforms were used to show that the eigenspaces of T are precisely the isotypic subspaces of such permutation modules. It also extends some of the ideas found in [10] and [11]. See [1, 2] for different examples of Radon transforms associated with buildings.

We assume the reader is familiar with buildings and chamber systems. More specifically, we assume the reader is familiar with the building of type $A_{n-1}(\mathbb{F}_q)$ whose chambers may be viewed as maximal flags in an n-dimensional vector space

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over the finite field $\mathbb{F}_q$ with q elements, and whose corresponding Weyl group is the symmetric group S_n (see, for example, [4, 13]).

2. BACKGROUND

In this section, we use incidence relations to define Radon transforms, and describe the computational model we use to compare different approaches to applying a fixed linear transformation. We also review some terminology related to buildings, and discuss the class sum of reflections in the finite general linear group.

Radon Transforms. Let G be a finite group acting on finite sets X and Y, and let M and N be the resulting $\mathbb{C}[G]$ permutation modules, respectively (see, for example, [14]). We say that the elements of X, when thought of as vectors in M, form the *usual basis* of M.

Suppose there is an incidence relation between X and Y. We write $x \sim y$ if $x \in X$ is incident to $y \in Y$, and define the *Radon transform* (see [3]) $R : M \to N$ by setting

$$R(x) = \sum_{y : x \sim y} y.$$

The adjoint $R^* : N \to M$ is defined by setting

$$R^*(y) = \sum_{x : x \sim y} x.$$

Thus, if the incidence relation is invariant under the action of G, then R, R^* and R^*R are $\mathbb{C}[G]$-homomorphisms.

Computational Model. As in [8], to compare different approaches to applying a fixed linear transformation, we use a computational model that counts one complex multiplication followed by one complex addition as one *operation*. Since a linear transformation T on a permutation module M may be viewed as a matrix with respect to the usual basis of M, the number of operations needed to apply T is never more than the number of nonzero entries in its corresponding matrix.

The Building of Type $A_{n-1}(\mathbb{F}_q)$. We now turn our attention to the action of the finite general linear group on the residues of the building of type $A_{n-1}(\mathbb{F}_q)$. See [4] or [13] for the relevant background on buildings. Note that, for convenience, we assume $q \neq 2$. The case $q = 2$ is, however, computationally similar. See [12] for details.

Let V be an $n > 1$ dimensional vector space over the finite field $\mathbb{F}_q$ with q elements, and let $GL_n(\mathbb{F}_q) = G_n$ be the group of automorphisms of V. Recall that the chambers of the building Δ of type $A_{n-1}(\mathbb{F}_q)$ may be viewed as nested sequences of subspaces, or *flags*,

$$V_0 \supset \cdots \supset V_n$$

where V_i is a subspace of V of dimension $n - i$. Let $I = \{1, \ldots, n-1\}$. If $i \in I$, then two chambers $V_0 \supset \cdots \supset V_n$ and $V_0' \supset \cdots \supset V_n'$ are said to be *i-adjacent* if $V_j = V_j'$ for all $j \neq i$. This gives rise to a chamber system over I. Residues of cotype $J = \{j_1, \ldots, j_{m-1}\} \subset I$ where $j_i < j_{i+1}$ may then be viewed as flags

$$V_0 \supset \cdots \supset V_m$$

where $V_0 = V$, $V_m = 0$, and V_i is a subspace of dimension $d_i = n - j_i$ when $0 < i < m$. For convenience, we say that the flag $V_0 \supset \cdots \supset V_m$ is of type $\lambda = (\lambda_1, \ldots, \lambda_m)$ where $\lambda_i = d_{i-1} - d_i$. For example, chambers correspond to flags of type $(1, \ldots, 1)$.

Let X_λ denote the set of flags of type λ. The action of G_n on V gives rise to a transitive action of G_n on X_λ, and we denote the corresponding $\mathbb{C}[G_n]$ permutation module by M_λ . Note that for any sequence $\mu = (\mu_1, \ldots, \mu_l)$ of non-negative integers whose sum is n, there is a corresponding set of flags X_μ of type μ and a permutation module M_μ.

If $b_1, \ldots, b_n$ are linearly independent vectors in V, let $(b_1, \ldots, b_n)$ denote the chamber

$$\langle b_1, \ldots, b_n \rangle \supset \langle b_2, \ldots, b_n \rangle \supset \cdots \supset \langle b_n \rangle \supset 0.$$

By fixing a basis $e_1, \ldots, e_n$ of V, we create the *fundamental apartment* Σ of Δ by taking nested sequences of subspaces spanned by subsets of $e_1, \ldots, e_n$. The chambers of Σ are therefore those maximal flags $(e_{\sigma(1)}, \ldots, e_{\sigma(n)})$ where σ ranges over the symmetric group S_n.

Recall that the Weyl group associated to Δ is also S_n and that we therefore have an *S_n-distance function*

$$\delta : \Delta \times \Delta \to S_n.$$

For example, if $x = (e_1, \ldots, e_n)$ and $y = (e_{\sigma(1)}, \ldots, e_{\sigma(n)})$, then $\delta(x, y) = \sigma$. Thus if the chambers x and y are i-adjacent, then $\delta(x, y)$ is the transposition $(i, i + 1)$.

The Class Sum of Reflections. The *fundamental reflections* of Σ are those automorphisms $s_1, \ldots, s_{n-1}$ where s_i exchanges the basis vectors e_i and e_{i+1} while leaving the other basis vectors fixed. The s_i are conjugate to each other in G_n and are therefore contained in the same conjugacy class C. Furthermore, each $c \in C$ fixes a *hyperplane* (codimension-1 subspace) of V pointwise. For example, s_1 fixes the hyperplane

$$\langle e_1 + e_2, e_3, e_4, \ldots, e_n \rangle$$

pointwise. We refer to C as the conjugacy class of *reflections* and denote the class sum of C by T.

Now if x is the (not necessarily maximal) flag $V_0 \supset \cdots \supset V_m$, then x determines a partition $\{P_1(x), \ldots, P_m(x)\}$ of the hyperplanes of V where we say the hyperplane H is in $P_j(x)$ if H contains V_j but not V_{j-1}. We will make use of this partition in this and the next section.

Let c be a reflection and suppose that c fixes the hyperplane $H \in P_j(x)$ pointwise. Fix $v \in V_{j-1} - (H \cap V_{j-1})$. The vectors v and cv are exchanged by c since c^2 is the identity. It follows that $v + cv \in H$, thus $cv = -v + h$ for some $h \in H$.

Suppose cv is contained in V_{i-1} but not V_i. Notice that $i \leq j$ and that each vector in $V_{j-1} - (H \cap V_{j-1})$ will yield the same i. Define $\varphi(x, c) = (i, j)$. It is easy to show that $x = cx$ if and only if $i = j$, and that if $i < j$, then $x \cap cx = (V_0 \cap cV_0) \supseteq \cdots \supseteq (V_m \cap cV_m)$ is a flag of type $\mu = (\mu_1, \ldots, \mu_m)$ where $\mu_i = \lambda_i + 1$, $\mu_j = \lambda_j - 1$, and $\mu_k = \lambda_k$ for $k \neq i, j$.

Thus, if $i < j$, we may define $T_{ij} : M_\lambda \to M_\lambda$ by setting $T_{ij}(x) = \sum y$ where the sum is over all y such that $y = cx$ for some reflection c where $\varphi(x, c) = (i, j)$.

If k is a non-negative integer, define $[k] = 1 + q + q^2 + \cdots + q^{k-1}$. We then have the following lemmas. Proofs may be found in [12].

Lemma 1. *Let* $x \in X_\lambda$*. If* $1 \leq i < j \leq m$*, then there are* $q^{2(d_i - d_{j-1})+1}[\lambda_i][\lambda_j]$ *flags* y *such that* $y = cx$ *for some reflection* c *where* $\varphi(x, c) = (i, j)$*.*

Lemma 2. *If* T *is viewed as a linear transformation* $T : M_\lambda \to M_\lambda$*, then*

$$T = \sum_{1 \leq i < j \leq m} q^{n-2+d_{j-1}-d_i}(q-1)T_{ij} + \left(\sum_{j=1}^{m} q^{n-1}[\lambda_j] \right) I.$$

3. Applying the Class Sum of Reflections

We now turn our attention to applying the class sum T of reflections to arbitrary vectors in M_λ. We begin by considering the direct application of T when viewed as a matrix with respect to the usual basis of M_λ.

Proposition 3. *The number of operations needed to apply* T *to an arbitrary vector in* M_λ *is no more than*

$$\left(1 + \sum_{1 \leq i < j \leq m} q^{2(d_i - d_{j-1})+1}[\lambda_i][\lambda_j] \right) |X_\lambda|.$$

Proof. When viewed as a matrix with respect to the usual basis of M_λ, each column of T contains

$$1 + \sum_{1 \leq i < j \leq m} q^{2(d_i - d_{j-1})+1}[\lambda_i][\lambda_j]$$

nonzero entries by Lemma 1 and Lemma 2. The proposition follows immediately. □

Using Radon Transforms to Apply T**.** As noted in the introduction, Proposition 3 may be improved upon by writing T as a linear combination of related Radon transforms.

Let x be the flag $V_0 \supset \cdots \supset V_m$ of type λ. Let $1 \leq i < j \leq m$ and let $\mu = (\mu_1, \ldots, \mu_m)$ where $\mu_i = \lambda_i + 1$, $\mu_j = \lambda_j - 1$, and $\mu_k = \lambda_k$ for $k \neq i, j$. We say that x is *ij-incident* to $y \in X_\mu$ if y is a flag of the form

$$V_0 \supset \cdots \supset V_{i-1} \supset H \cap V_i \supset \cdots \supset H \cap V_{j-1} \supseteq V_j \supset \cdots \supset V_m$$

for some $H \in P_j(x)$. This incidence relation is invariant under the action of G_n, thus the associated Radon transform $R_{ij} : M_\lambda \to M_\mu$ is a $\mathbb{C}[G_n]$-homomorphism.

The following is Theorem 13 in [12]:

Theorem 4. *If* T *is viewed as a linear transformation* $T : M_\lambda \to M_\lambda$*, then*

$$T = \sum_{1 \leq i < j \leq m} q^{n-1+d_{j-1}-d_i-i}(q-1)R_{ij}^* R_{ij} + \left(\sum_{j=1}^{m} q^{n-j}[\lambda_j] \right) I.$$

We may therefore apply T using the R_{ij} and R_{ij}^*.

Lemma 5. *The number of operations needed to apply* R_{ij} *or* R_{ij}^* *is no more than*

$$q^{d_i - d_{j-1}}[\lambda_j]|X_\lambda|.$$

Proof. By Lemma 5 in [12], each flag in M_λ is ij-incident to $q^{d_i - d_{j-1}}[\lambda_j]$ flags in M_μ. Thus, when viewed as a matrix with respect to the usual basis, each column of R_{ij} contains $q^{d_i - d_{j-1}}[\lambda_j]$ nonzero entries. Since R_{ij}^* is the transpose of R_{ij}, the lemma follows. □

Given Lemma 5, we may now state the following theorem, which should be compared to Proposition 3:

Theorem 6. *The number of operations needed to apply T to an arbitrary vector in M_λ is no more than*

$$\left(\frac{m^2-m+2}{2}+\sum_{1\le i<j\le m} 2q^{d_i-d_{j-1}}[\lambda_j]\right)|X_\lambda|.$$

Proof. By Lemma 5, the number of operations needed to apply a scalar multiple of $R^*_{ij}R_{ij}$ is no more than

$$(1+2q^{d_i-d_{j-1}}[\lambda_j])|X_\lambda|.$$

Thus, by Theorem 4, the number of operations needed to apply T is no more than

$$\left(1+\sum_{1\le i<j\le m}(1+2q^{d_i-d_{j-1}}[\lambda_j])\right)|X_\lambda| = \\ \left(\frac{m^2-m+2}{2}+\sum_{1\le i<j\le m} 2q^{d_i-d_{j-1}}[\lambda_j]\right)|X_\lambda|.$$

□

Restricting to Chambers. We now restrict our attention to the action of G_n on the chambers of Δ. For convenience, let X denote the set $X_{(1,\dots,1)}$ of chambers and let M denote the resulting $\mathbb{C}[G_n]$-module $M_{(1,\dots,1)}$.

When viewed as a matrix with respect to the usual basis of M, the number of operations needed to directly apply T to an arbitrary vector in M is no more than

$$\left(1+\sum_{1\le i<j\le n} q^{2(j-i)-1}\right)|X|$$

by Proposition 3. By Theorem 6, this number may be improved to

$$\left(\frac{n^2-n+2}{2}+\sum_{1\le i<j\le n} 2q^{(j-i)-1}\right)|X| \tag{1}$$

by using Radon transforms. This bound may be improved even further by taking into account the relationship between reduced galleries and the S_n-distance function δ.

Lemma 7. *Let c be a reflection and let x be a chamber. If $x\ne cx$ and $\varphi(x,c)=(i,j)$, then $\delta(x,cx)$ is the transposition $(i,j)\in S_n$.*

Proof. Let H be the hyperplane fixed pointwise by c. It is easy to show that if $\varphi(x,c)=(i,j)$ and $i\ne j$, then there is a basis $b_1,\dots,b_n$ of V such that $x=(b_1,\dots,b_n)$, c exchanges b_i and b_j, and $b_k\in H$ if $k\ne i,j$. It follows that $\delta(x,cx)$ is the transposition (i,j). □

We may now take advantage of the relationship between reduced galleries in Δ and the S_n-distance function $\delta:\Delta\times\Delta\to S_n$ to show how T may be applied with a surprisingly small number of operations. In particular, the following bound replaces the scalar $(n^2-n+2)/2+\sum_{1\le i<j\le n}2q^{(j-i)-1}$ in (1) with the much smaller n^3.

Theorem 8. *The number of operations required to apply* $T : M \to M$ *is less than*

$$n^3|X|.$$

Proof. By Lemma 7, $T_{ij} : M \to M$ may now be defined by setting $T_{ij}(x) = \sum y$ where the sum is over all chambers y such that $\delta(x,y) = (i,j)$. The transposition (i,j) may be written as a reduced product of adjacent transpositions:

$$(i,j) = (i,i+1)\cdots(j-2,j-1)(j-1,j)(j-2,j-1)\cdots(i,i+1).$$

It follows that

$$T_{ij} = T_{i,i+1}\cdots T_{j-2,j-1}T_{j-1,j}T_{j-2,j-1}\cdots T_{i,i+1}.$$

By Lemma 12 in [12], $T_{i,i+1} = R^*_{i,i+1}R_{i,i+1} - I$. Thus, by Lemma 5, $T_{i,i+1}$ may be applied using no more than $3|X|$ operations. A scalar multiple of T_{ij} may therefore be applied using no more than

$$|X| + (2(j-i)-1)(3|X|) = (6(j-i)-2)|X|$$

operations. Hence, by Lemma 2, the number of operations required to apply T is no more than

$$\left(1 + \sum_{1\le i<j\le n} (6(j-i)-2)\right)|X| < n^3|X|.$$

□

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