Math 12: Discrete Dynamical Systems Homework

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Several of these problems will require computational software to help build our insight about discrete dynamical systems. On the web, there's a collection of applets by Kurt Cogswell at:

http://learn.sdstate.edu/cogswelk/ChaosFractals/Applets_Page.htm

The *Web Diagram Applet* on this page will be the one that is most relevant for our purposes.

In the PC Computer Labs, there is the more sophisticated *Discrete Tool* in *ODE Architect*. (*ODE Architect* is software that normally comes bundled with the differential equations textbook by Borelli and Coleman).

In most software, remember that multiplication in equations must be denoted by a $*$.

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1. Use *graphical analysis* to analyze the dynamics of the linear recurrence

$$x_{n+1} = ax_n + b$$

in each of the following cases. Determine the long-term behavior of the system for all possible initial states x_0. (Are there fixed points? Periodic points? If so, find them. Do some orbits fly off to ∞ or $-\infty$? Do some orbits approach fixed points?)

When completed, check your answers using dynamical systems software (e.g., *Discrete Tool*, *Web Diagram Applet*).

(a) $a = 1, b > 0$.

(b) $a = -1$ and b any real number.

2. Use dynamical systems software to answer the following questions. Let x_n = proportion of sick people each month n. The *logistic map* is a model for the spread of infection:

$$x_{n+1} = cx_n(1 - x_n).$$

The parameter c may be thought of as the transmission rate. (In *Discrete Tool*, this map is loaded up by default... just change the value of c to what you want. In *Web Diagram Applet*, choose "Logistic Parametrized Family" from the menu.)

(a) Let $c = 2.9$. Choose any initial seed x_0 between 0 and 1.
Give BRIEF answers for the following questions.
Can you determine the long-term behavior of its iterates, and if so, what happens?
Give numerical estimates for any limiting values you detect.
Try several different seeds... does the long-term behavior change?
What do your results mean in terms of the spread of infection?

(b) Now let $c = 3.832$. Answer the same questions as in (a) above.

(c) Now let $c = 3.8$. Answer the same questions as in (a) above.

3. In this problem, you will prove the Mean Value Theorem in a series of guided steps. Let $a \neq b$. The Mean Value Theorem says: if $g : [a, b] \to \mathbf{R}$ is differentiable, then there exists a $c \in (a, b)$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}. \tag{1}$$

To prove this, assume g is differentiable as given, and perform the following steps:

(a) What is the equation of the secant line through $(a, g(a))$ and $(b, g(b))$?

(b) Construct a new function f from g by setting

$$f(x) = g(x) - \left[g(a) + \frac{g(b) - g(a)}{b - a}(x - a) \right].$$

(Notice where the variable x appears. Everything else are just constants!) What does the bracketed term have to do with the secant line from part (a)?

(c) Evaluate $f(a)$, $f(b)$, and $f'(x)$.

(d) Can Rolle's theorem be applied to f? (Rolle's Theorem says: if $f : [a, b] \to \mathbf{R}$ is differentiable, and $f(a) = f(b) = 0$, then there exists a c in (a, b) such that $f'(c) = 0$.) Use this information to derive Equation (1). This proves the Mean Value Theorem.

4. Find the fixed points for all of the following discrete dynamical systems $x_{n+1} = f(x_n)$. Use the Multiplier Theorem to determine the *stability* of the fixed points (are they stable, unstable, or neither?). If the Multiplier Theorem is inconclusive, use graphical analysis to determine your answer.

(a) $f(x) = 1/(x - 1)$.

(b) $f(x) = x^2 - 2$.

(c) $f(x) = x^{2/3}$.

(d) $f(x) = \sin(x)$. [Hint: draw a graph].

5. Finding the fixed points of the dynamical system generated by $f(x) = \cos(x)$ is impossible to do analytically, since solving $x = \cos(x)$ in closed form is impossible. However, in this problem you will discover how to locate them anyway.

(a) Can integer multiples of $\frac{\pi}{2}$ be fixed points of this system?

(b) If f had a fixed point p, what would the Multiplier Theorem say as long as p was not a integer multiple of $\frac{\pi}{2}$?

(c) Thus, if p exists, you should be able to find it by iterating f with an appropriate seed. Start with $x_0 = 1$. You can easily iterate this function by pressing the Cosine button on your calculator! What is the numerical value of the fixed point, to 3 decimal places? How many iterations did it take?

(d) Does every initial seed x_0 approach this fixed point? Use graphical analysis to get some intuition, then justify your answer. [Hint: for any x_0, where is $x_1 = f(x_0)$?]

6. Is it possible for a 1-dimensional linear dynamical system to have exactly 2 fixed points? Justify.

7. Recall the *logistic map* $f(x) = \lambda x(1 - x)$, that models the spread of infection among a population. Let $\lambda = 1$. Find the fixed point of this map. Does the Multiplier Theorem say whether this is a stable fixed point? If not, use graphical analysis to say whether this fixed point is stable, unstable, or stable on one side but not the other.

8. Consider the function $f(\theta) = 3\theta$ where θ is an *angle*. (Thus θ may take on only values in the range $[0, 2\pi)$. For example, $f(3\pi/4) = \pi/4$, since $9\pi/4 = \pi/4$ for angles.)

   Find all the fixed points of the dynamical system $\theta_{k+1} = f(\theta_k)$.

9. Consider the dynamical system given by iteration of the map

$$f(x) = -\frac{1}{2}(x^3 + x).$$

   (a) Both 1 and -1 are periodic points of period 2, and part of the same orbit. Use dynamical systems software to guess whether this is a stable periodic orbit, and then prove your guess using the Multiplier Theorem for periodic orbits.

   (b) Plot $f(f(x))$ in Maple and use this plot to determine if there are any *other* periodic points of f of *prime* period 2.

   (To define this function f in Maple, use the command

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f := x -> -(x^3+x)/2;
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   . Then you can plot $y = f(f(x))$ and the line $y = x$ by a command like:

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plot( { f(f(x)), x } , x=-2..2, y=-2..2);
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   You can change the range of x and y to investigate where the lines intersect.)

10. Challenge Question: [if you do it, turn in to your professor] Develop a higher derivative test for determining whether a fixed point is stable or not in cases where the Multiplier Theorem is inconclusive. [Hint: use Taylor's Theorem with the derivative form of the remainder.]

11. As an environmental engineer, you have been asked to analyze the dumping of a pollutant in a nearby lake, and find that the amount of pollution entering the lake from all sources is about 20 pounds per year. Fortunately, sunlight gradually breaks down the pollutant into harmless by-products; each year, 10% of the pollutant is lost by this process.

    Currently in the lake, there are 0.2 pounds of pollutant per cubic mile of water. Safe levels have been defined by the EPA to be 0.1 pounds per cubic mile. The lake contains 1500 cubic miles of water.

    In order to simplify the situation, we assume that the 20 pounds of pollutant all gets added at the end of the year. The dynamical system which models the above process is given by

$$x_{n+1} = 0.9x_n + 20.$$

    (a) Will the lake ever reach safe levels? Determine your answer by using a fixed point and stability analysis.

    (b) Now suppose you had a method to remove more pollution so that 20% of the pollutant was removed each year. Will the lake ever reach safe levels? Justify. If so, use software (or a programmable calculator) to determine how many years it takes.

    (c) Suppose that you were wrong about your model and in fact the amount of pollutant that sunlight breaks down is not 10% or 20% but is $1.64\sqrt{x_n}$ pounds per year. Use software to determine whether the lake will ever reach safe levels. If so, how many years will it take?

12. Suppose you would like to find the roots of $f(x)$. In Newton's method, one iterates the function

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

(a) Show that if p is a root of f, then p is a fixed point for g.

(b) We know that iteration of g seems to converge as long as the initial seed x_0 is "close enough" to a root of f. In other words, p appears to be an *attracting* fixed point of g when p is a root of f. Use the Multiplier Theorem on g to show that this is true.

13. Consider the logistic map $f(x) = \lambda x(1 - x)$ when $\lambda = 3.8319$. A period three orbit for this map is (approximately) $\{0.5, 0.948, 0.1543\}$. Use the Multiplier Theorem for periodic orbits to determine whether this periodic orbit is stable or not.

14. Let $f(x) = -\frac{13}{12}x^2 + \frac{49}{12}x + 1$.

Verify that $x = 4$ is a periodic point. What is its prime period? What are the other points in its orbit?

Use the multiplier theorem to determine if this a stable or unstable periodic orbit.

15. Define the following dynamical system: let the state space $S = [0, 1]$, all numbers between 0 and 1. Let f be the function which takes the decimal representation, cuts off the first digit, and shifts everything left. [Example: f(0.1415926...)= (.415926...)].

(a) What are all the fixed points of f? (In other words, which numbers which remain unchanged after applying f once?)

(b) Describe all the periodic points of f of prime period 2?

(c) Describe all points of f which are *eventually periodic* of any period? (Eventually periodic means that the iterates eventually repeat, though they may or may not repeat at first.)

16. Graph the following functions for a range of values of the parameter c to determine whether the bifurcation that occurs is a tangent, or period-doubling bifurcation. Using software estimate the precise value of c in each case where the bifurcation occurs.

(a) $f(x) = ce^x$, as c varies between $c = .3$ and $c = .4$.

(b) $f(x) = x^2 - c$, as c varies between $c = .6$ and $c = .9$. [Focus on the fixed point that is smaller and negative.]

17. Water is dripping from a leaky faucet. The time between drips can be modeled by some (unknown) discrete dynamical system. There is a parameter λ associated with a dynamical system which is the tightness (marked by the number of turns) of the faucet handle. At first, you observe that the time between drips is constant at 1 second between drops. As you unscrew the faucet handle past the 1/4-turn mark, the drip times begin to exhibit period 2 behavior (e.g., .7, .9, .7, .9,... seconds, etc.)

We say that there is a *bifurcation* at $\lambda_1 = 0.25$ turns.

Then as you unscrew the handle past the 1/2-turn mark, it begins to exhibit period 4 behavior. Hence we'll say $\lambda_2 = 0.5$ turns.

Although you don't know the underlying dynamical system, you can still make predictions! Using Feigenbaum's universal constant, *estimate* λ_3, the position of the handle at which the next period-doubling will occur.

Now try this with a real faucet and see if you can observe period-doubling.

18. Which of the following exhibit sensitive dependence on initial conditions? (Give BRIEF justifications.)

(a) $f(x) = \frac{1}{2}x(1-x)$ on $[0,1]$

(b) $f(x) = x/3$ on $\mathbf{R}$

(c) $f(x) = 5x$ on $\mathbf{R}$

(d) $f(x) = x + 5$ on $\mathbf{R}$.

19. Which of the following are *dense* sets on $[0,1]$?

(a) the set $\{0.1, 0.5, 0.8\}$?

(b) the rational numbers in $[0,1]$.

(c) $\{1/2, 1/4, 1/8, 1/16, ...\}$.

(d) *any* fraction in $[0,1]$ with denominator 2^i for some integer i.

20. Consider $f(x) = \lambda x(1-x)$, the dynamic that models the spread of infection. Let x_n represent the fraction of the population infected at year n.

(a) Let $\lambda = 3.7$. Suppose you measure the fraction of infected people at the start to be 0.33 instead of the true value 0.333. Calculate the orbits of $x_0 = 0.33$ and $x_0 = 0.333$. After how many years will the two orbits diverge enough that your prediction will be off by 20% of the population?

(b) Now let $\lambda = 3.5$. Calculate the orbits of $x_0 = 0.33$ and $x_0 = 0.333$ again. Do the orbits ever separate by more than 20% of the population?

(c) In which of the above situations can you trust your prediction? Based on these examples, why is it important to know whether a dynamical system has sensitive dependence on initial conditions?

21. Recall the *shift map* on decimal numbers between 0 and 1, where f is the function that takes the decimal representation, cuts off the first digit, and shifts everything left. [Example: f(0.1415926...)= (.415926...)]. As it turns out, f is a chaotic dynamical system. In this exercise, you will verify that the three conditions of a chaotic system hold for f in special cases.

(a) If $\delta = .01$, show that the points $x = 0.1125, y = .1126$ eventually separate by more than δ under iteration by f, by finding the smallest k such that $f^k(x)$ and $f^k(y)$ are more than .01 apart.

Now consider the points $x = 0.112599$ and $y = 0.112601$ and determine the smallest k such that $f^k(x)$ and $f^k(y)$ are more than .01 apart.

(b) Find an initial seed x_0 in the interval $U = (0.1125, 0.1126)$ that visits the interval $(0.777, 0.778)$ after exactly 4 iterations.

(c) Find a periodic point in the interval $(0.1125, 0.1126)$.

22. You can perform the shift map on $[0,1]$ on your calculator by multiplying the given number by 10, and removing the first digit. Start with the initial seed $x_0 = 1/\pi$, and iterate over and over. After several iterations, you will get 0. This is incorrect. Explain why your calculator made this mistake.

23. In this problem, you are asked to prove special cases of Sarkovski's theorem.

(a) Show that if a continuous function f has a periodic point of period 2, then it must have a fixed point.

(b) Show that if a continuous function f has a periodic point of period 4, then it must have a fixed point. [Hint: think about f^2.]

24. Use Sarkovskii's theorem to order the following numbers from left to right, such that if i is to the left of j, then any function with a period i point implies there exists a period j point as well.

$2, 3, 4, 5, 6, 13, 16, 20.$

25. (From Lay's *Linear Algebra and its Applications*) In the redwood forest of California, wood rats are eaten by the spotted owl. Let O_k, R_k denote the populations of the owls and rats at time k (in months). Suppose that

$$O_{k+1} = .5O_k + .4R_k$$

$$R_{k+1} = -.104O_k + 1.1R_k.$$

Then if $\mathbf{x}_k = \binom{O_k}{R_k}$, we can represent the dynamics of these populations as $\mathbf{x}_{k+1} = A\mathbf{x}_k$, a 2-dimensional dynamical system.

(a) Verify that the eigenvalues of A are $\lambda_1 = 1.02$ and $\lambda_2 = .58$, and the corresponding eigenvectors are $\binom{10}{13}$ and $\binom{5}{1}$.

(b) Suppose that the initial populations are $O_0 = 20$ and $R_0 = 15$. Find an explicit expression for the populations of O_n and R_n at time n. What is the long-term ratio of the population of owls to rabbits in this case?

(c) Suppose that the initial populations are $O_0 = 10$ and $R_0 = 2$. What is the long-term behavior of the populations of owls and rabbits in this case?

(d) If both eigenvalues had been smaller than 1 in magnitude, what could you conclude about the long-term behavior of rabbit and owl populations?