Math 132 — HW I

The first problem is a famous theorem! Doing it will help you review concepts from Analysis I.

I.) *The contraction mapping theorem.*

Let X be a metric space, with a metric d. If $\phi : X \to X$, and if there exists a number $c < 1$ such that for all $x, y \in X$

$$d(\phi(x), \phi(y)) \ < \ c\ d(x, y)\ ,$$

then ϕ is called a *contraction* of X into itself.

Prove:

(a) that ϕ is continuous

(b) that if X is a *complete* metric space, and ϕ is a contraction of X into itself, then there exists a *unique* fixed point $x \in X$ such that $\phi(x) = x$.

In the next two problems you will prove some inequalities that are very useful in practice!

II.) Use the MVT to show that $e^x \geq 1 + x$ for all $x \in \mathbf{R}$. (You may assume knowledge of the derivative of e^x.)

III.) Show that $1 - \frac{x^2}{2} \leq \cos(x)$ for all $x \in \mathbf{R}$. (You may assume knowledge of the derivatives of sine and cosine.)

Do also:

Chapter 5 (Problem 25, all parts) and interpret the result of part (d): what is it saying that is so interesting?