PCMI Undergraduate Faculty Program

Facilitated by:
Francis Edward Su (organizer)
with assistance from Daniel Goroff (director of the UFP)

The theme of this year's UFP is: *Combinatorics in Concert: for Teaching, Research, Outreach, and Recreation*. The program will consists of two parts:

* **A Baseline:** [morning lectures, daily]— A survey of topics in geometric combinatorics

* **A Melody:** [afternoon sessions, daily except Wed.]— Assorted guest talks or discussion related to aspects of a faculty member's professional life

Participants from other programs at PCMI are welcome to attend any of our activities.

Baseline Lectures: a survey of topics in geometric combinatorics

There are many topics that could be discussed in a course on geometric combinatorics; but in these lectures I have chosen what I consider to be my favorite topics for inclusion in such a course. Mostly the lectures reflect what I like in the subject. Some of the topics are of relatively recent development and reflect either my current interests (e.g., combinatorial fixed point theorems) or material I assimilated in the "Discrete and Computational Geometry" program at MSRI in Fall 2003. I will also give examples of research projects that I have given undergraduates in this area. The goals of the lectures are:

- To introduce one to a selection of topics in geometric combinatorics
- To equip one to attend some of the advanced lectures in the PCMI Graduate Summer School or Research Program, and to begin reading literature in the area
- To enrich one's teaching with new examples or new course ideas
- To stimulate ideas for research problems for oneself or one's students

Melody Seminar: professional development

We will focus on several aspects of the faculty member's professional life, such as:

- Becoming more informed about educational issues and teaching methods
- Involving undergraduates in research
- Enriching one's own teaching with examples from geometric combinatorics
- Doing research in this area
- Generating interest in math by hosting problem-solving sessions or other activities
- Preparing outreach lectures to the community

UFP Participants will design projects related to some of these topics and make presentations on the last day.

Tentative Schedule for the UFP

Note that all Morning Lectures (the "Baseline" survey course) are at 11:00am.
All Afternoon Seminars (the "Melody") are at 1:00pm, unless otherwise noted.
Please check the daily schedule for updates.

Week 1

- Mon July 12:
  Morning Lecture – *Combinatorial Convexity*
  Afternoon Seminar – Welcome and Introduction to the UFP

- Tue July 13:
  Morning Lecture – *Helly's theorem and its Relatives*
  3:15pm Afternoon Seminar – Tom Roby, "Mathematicians working with teachers in California: The ACCLAIM Experience"

- Wed July 14:
  Morning Lecture – *Polytope Basics I: Examples and Construction*

- Thu July 15:
  Morning Lecture – *Polytope Basics II: Duality*
  Afternoon Seminar – Tom Roby, "Japanese Lesson Study: teaching cultures in Japan and California"

  6:00pm – Pizza and Problem-Solving

- Fri July 16:
  Morning Lecture – *Polytope Basics III: Combinatorics of Faces*
  4:30pm Afternoon Seminar – Richard Hill, "What in High School math courses promote, or does not promote, success in university math courses?"

Week 2

- Mon July 19:
  Morning Lecture – *An Introduction to Phylogenetic Trees*
  Afternoon Seminar – "Discussion: Planning an Outreach Lecture"

- Tue July 20:
  Morning Lecture – *An Introduction to Tropical Geometry*
  Afternoon Seminar – David Perkinson, "Undergraduate Research Projects on Polytopes"

- Wed July 21:
  Morning Lecture – *Combinatorics of Triangulations*

- Thu July 22:
  Morning Lecture – *Minkowski's Theorem*
  Afternoon Seminar – "Discussion: Planning Course Modules"

  6:00pm – Pizza and Problem-Solving

- Fri July 23:
  Morning Lecture – *What are Ehrhart Polynomials?*
  Afternoon Seminar – Francis Su, "Running a Problem-Solving Seminar and Encouraging Math Community"

Week 3

- Mon July 26:
  Morning Lecture – *Combinatorial Fixed Point Theorems I : Sperner's Lemma*
  Afternoon Seminar – Dan Schaal, "Undergraduate research: how to get it started and keep it going"

- Tue July 27:
  Morning Lecture – *Combinatorial Fixed Point Theorems II : Tucker's Lemma*
  Afternoon Seminar – Dan Schaal, "Undergraduate research: how to find problems and what to do when you solve one"

- Wed July 28:
  Morning Lecture – *Combinatorial Fixed Point Theorems III : Kneser Colorings*

- Thu July 29:
  Morning Lecture – *Combinatorial Fixed Point Theorems IV: Trees and Graphs*
  Afternoon Seminar – "Discussion: Geometric Combinatorics in the Curriculum"

- Fri July 30:
  Morning Seminar – *Presentations by UFP Participants*
  Afternoon Seminar – *Presentations by UFP Participants*

For Fun— Some Open Problems

1. Throw k points down in the unit square and find the area of the largest convex set in the square containing none of the k points. Let $f(k)$ be the minimum (of the largest areas) over all sets of k points. Find good upper and lower bounds on $f(k)$.

   (a problem of Moser, as told by Lay [6, p.92] For $k = 3$ it is known that $1/3 \le f(3) \le \sqrt{2}/4$.)

2. What is the largest area that a n-gon of unit diameter can have?

   (A problem discussed in [6, p.92]. For odd n, the polygon must be regular. For $n = 4$ there are infinitely many quadrilaterals. For $n = 6$, Graham [5] shows that there is a non-regular hexagon that achieves maximum area, and it is unique.)

3. A Radon relative: can every set of 8 points in the plane be partitioned to form 2 triangles and a line segment so that the segment cuts the interior of both triangles?

   (A problem discussed in [2, p.137], related to Tverberg's generalization of Radon's lemma)

4. Let $\mathcal{A}$ be a finite family of translates of a convex set in $\mathbb{R}^2$. Prove or disprove that if every two members of $\mathcal{A}$ intersect, then some set of 3 points intersects every member of $\mathcal{A}$.

   (from the Wenger article in [4, p.71]. There are many more Helly-type open problems mentioned in the Wenger article.)

5. Let P be a d-polytope, and let f_i be the number of faces of dimension i. Prove that $f_i \ge \min\{f_0, f_{d-1}\}$.

   (a problem of I. Barany, as told by G. Kalai in a Fall 2003 MSRI workshop)

6. Is the graph of triangulations of every 2-dimensional point set $(n-3)$-connected?

   (from Aug 2003 MSRI workshop on triangulations, DeLoera et al. It is known all triangulations have at least $n-3$ flips.)

7. Let A be a set of n points in the plane, and consider all triangulations of A that use all the vertices. Find a good upper bound of the form C^n.

   (from Aug 2003 MSRI workshop on triangulations, DeLoera et al. The best known upper bound is 59^n. See [7]. It is believed that there are at most 8^n of them.)

8. The Perron-Frobenius theorem says that if A is a non-negative $n \times n$ matrix, then A has a non-negative eigenvalue. One proof of this theorem uses the Brouwer fixed point theorem. Find a proof of this theorem that directly uses Sperner's lemma.

9. Chamber complexes of polytopes are not well understood at all. Study the chamber complex of any polytope and try to explain its structure. For instance, what about the prism over a regular triangle, or any prism over a regular polygon? Can you enumerate the number of chambers? (Some related references: [1, 3].)

10. A generalization of the Borsuk-Ulam theorem due to Yang [8, 9] says: for every continuous function $S^{dn} \to \mathbb{R}^d$, there exist n mutually orthogonal diameters whose $2n$ endpoints are mapped to the same point. A theorem of Kakutani-Yamabe-Yujobo says: for every continuous function $S^n \to \mathbb{R}$, there exist $n+1$ mutually orthogonal radii whose $n+1$ endpoints are mapped to the same point. Find a combinatorial analogue for either of these theorems. (Start with small n and d).

References

[1] Tatiana V. Alekseyevskaya. Combinatorial bases in systems of simplices and chambers. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 15–37, 1996.

[2] Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy. *Unsolved problems in geometry*. Problem Books in Mathematics. Springer-Verlag, New York, 1994. Corrected reprint of the 1991 original. Unsolved Problems in Intuitive Mathematics, II.

[3] Jesus A. De Loera, Elisha Peterson, and Francis Edward Su. A polytopal generalization of Sperner's lemma. *J. Combin. Theory Ser. A*, 100(1):1–26, 2002.

[4] Jacob E. Goodman and Joseph O'Rourke, editors. *Handbook of discrete and computational geometry*. CRC Press Series on Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, 1997.

[5] R. L. Graham. The largest small hexagon. *J. Combinatorial Theory Ser. A*, 18:165–170, 1975.

[6] Steven R. Lay. *Convex sets and their applications*. Robert E. Krieger Publishing Co. Inc., Malabar, FL, 1992. Revised reprint of the 1982 original.

[7] Francisco Santos and Raimund Seidel. A better upper bound on the number of triangulations of a planar point set. *J. Combin. Theory Ser. A*, 102(1):186–193, 2003.

[8] Chung-Tao Yang. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. I. *Ann. of Math. (2)*, 60:262–282, 1954.

[9] Chung-Tao Yang. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. II. *Ann. of Math. (2)*, 62:271–283, 1955.

(My) Principles for Undergraduate Research

My philosophy about undergraduate research:

- Undergraduates can do research, and often, publishable research.
- For the undergraduate, the process is more important than the outcome.
- My role as advisor is not to be the expert in all things, but to show the student how to become an expert in one thing.
- If one understands a problem well enough, eventually one will see something new.
- If one tries to prove a theorem without looking at the original proof, one will often find a new proof.
- Clear thinking is equivalent to clear writing.

About choosing problems:

- Pick problems with small, do-able steps.
- Modify a known problem by changing the hypothesis, changing the proof method, or changing the context.
- Try to make connections between areas that haven't been connected before.

About helping students succeed:

- Build confidence by giving the student do-able goals for each meeting. (And then expect them to meet those goals.)
- Give the student some ownership of the research direction. What questions about the given problem does she find interesting?
- When reading related work, encourage students to think before they read— to discover ideas for themselves before seeing how others have done it.
- Encourage the student to always be writing up (LaTeX) their work. Require a written final report at the end of the project period. The last 2 weeks should be devoted exclusively to writing.
- Give constant reassurance.
- Let the students do the talking. They'll learn how to explain their ideas to others. Encourage them to talk to each other.