

SOME PREPARATORY PROBLEMS TO PONDER IN ADVANCE

1. The *median* of a data set of real numbers a number for which half the data lies on either side. But what if the data set are points in the plane? Can one define a kind of “median”? For instance, is there always a point in the plane for which any line through that point cuts the data set in half? Is there always a point so that any line through that point has at least $1/3$ of the points on each side? Try some examples.
2. An art gallery is in a polygonal room (with finitely many sides, and possibly non-convex) and there is one painting hung on each wall. Suppose that for any 3 paintings, there is a point in the gallery where those 3 paintings are visible. Is there always a point of the gallery from which all paintings are visible?
3. Can a convex bounded polyhedron have seven edges?
4. Let S be the square $[-1, 1] \times [-1, 1]$. Let S^Δ be the set of all points in the plane whose dot product with any point in S is less than or equal to 1: $S^\Delta = \{\mathbf{y} \in \mathbf{R}^2 : \mathbf{y} \cdot \mathbf{x} \leq 1 \ \forall \mathbf{x} \in S\}$. Find S^Δ . Find $(S^\Delta)^\Delta$. See what this operation does to other sets S .
5. Take an n -gon and subdivide it into triangles whose vertices are chosen from the vertices of the n -gon. In this subdivision, how many triangles are needed to cover the n -gon? And in how many ways can this subdivision be done? (Find a formula in terms of n .)
6. Suppose you try to subdivide a cube into several tetrahedra, each of whose vertices are vertices of the cube. What is the minimum and maximum number of tetrahedra that will accomplish this? Prove your claims.
7. A cake (with some goodies on it) is divided by parallel knives into 3 pieces from left to right. This division can be represented by a vector (x_1, x_2, x_3) where x_i is the width of the i -th piece. Thus $x_i \geq 0$ and $\sum x_i = 1$, so that the set of all divisions forms a triangle (the piece of the plane $\sum x_i = 1$ in the first octant). Make whatever assumptions seem reasonable for your preferences (note that your preferences don't have to be related to the width of the pieces). Let C_i be the set of divisions on which you will prefer to take piece i . What properties will the sets C_i satisfy? Must all the sets have a common point? Investigate the analogous problem for n pieces.
8. Suppose you have graph formed by an $m \times n$ grid of squares, with each node (the places where the corners meet) labelled by either $+1$, -1 , $+2$, or -2 . Suppose there are boundary conditions: the nodes on the top (respectively, bottom) side of the graph are labelled $+1$ (respectively -1), and all remaining nodes on the right side (respectively, left side) are labelled $+2$ (respectively -2). Show that there must either be (i) an edge in the graph whose two labels sum to zero, or (ii) a square in the grid with all four labels.
9. Suppose a convex n -gon in the plane P has integer vertex coordinates. Let $D(n)$ be the diameter of the smallest such n -gon. Try to find upper and lower bounds for $D(n)$.
10. Let $a \oplus b = \min\{a, b\}$ and $a \otimes b = a + b$. Make sense of these vector operations: what is $(2, 3, 3) \oplus (4, 6, 9)$? What is $3 \otimes (2, 3, 3) \oplus 0 \otimes (4, 6, 9)$? What are all the “convex” combinations of $(2, 3, 3)$ and $(4, 6, 9)$?