This Is Like That: Counting Using Catalan Numbers

Pallavi Jayawant (Bates College), Dan Schaal (South Dakota State University), Jim Seibert (Regis University), Jessica Sidman (Mount Holyoke College), Kevin Woods (University of California, Berkeley)

Abstract

This course module is for a Discrete Math or Combinatorics course in which Catalan numbers have already been introduced. We assume students have been led to a formula for Catalan numbers, probably through inclusion/exclusion counting of stairstep paths from $(0, 0)$ to (n, n) that remain below the diagonal, and we offer a plan to guide students to finding bijections between various objects in counting problems that also utilize Catalan numbers.

Introduction and Goals

Our goal in this module is to look at a different example of a counting problem that uses Catalan numbers, and to guide the students in class toward discovering a bijection that allows them to count something new. In the exercises they will be presented with different opportunities to use Catalan numbers in different situations.

To start, remind the students of the formula for Catalan numbers

$$C_n = \frac{(2n)!}{n!n!(n+1)} = \frac{\binom{2n}{n}}{n+1},$$

and that this counts the number of stairstep paths from $(0, 0)$ to (n, n) that remain below the diagonal. Have the students compute the first few Catalan numbers, starting with C_0. (The first few are $1, 1, 2, 5, 14, 42$. Write them on the board for the students to refer to later.)

The Mother of All Examples

You can motivate the following example by mentioning that it arises in computer science. Computers can only multiply two terms at a time and need to parenthesize a larger product before proceeding. We want to count the number of ways to parenthesize a non-associative product of n factors into pairs. Call this number P_n.

You can quickly compute the number of ways of parenthesizing 1, 2 and 3 factors

$$\begin{aligned} (a):&\quad P_1 = 1 \\ (ab):&\quad P_2 = 1 \\ ((ab)c) \quad (a(bc)):&\quad P_3 = 2 \end{aligned}$$

At this point have the students spend a few minutes finding all the ways to parenthesize four factors. Encourage them to be systematic so that they are sure they find all the ways. The list should consist of

$$((ab)(cd)), \quad (((ab)c)d), \quad ((a(bc))d), \quad (a((bc)d)), \quad (a(b(cd)))$$

If anyone finishes quickly, you might have them look at 5 factors and see if they find all 14.

Hopefully, the students will notice the list of Catalan numbers still on the board and see the connection between P_n and C_{n-1}. Now that they think they know the answer, ask them how they intend to prove it. Point out that mathematicians are lazy, and that we don't want to repeat all the work we did to find the formula for C_n.

Guide them through discussion into looking for a bijection between the stairstep paths they counted earlier and the ways of parenthesizing. Look at the $n = 4$ case to start guessing the bijection (drawing the stairstep paths on the board will help).

The easy direction is to start with the parenthesized products and try to match them to the stairstep paths. The first guess that is likely to come up is that "(" corresponds to "right" and ")" corresponds to "up". Get them to think about why this fails. A bijection that works is "(" corresponds to "right" and any letter (through the $(n-1)^{\text{st}}$) corresponds to "up". Ask them why the path is always below the diagonal, and why the path goes to $(n-1, n-1)$.

To go in the other direction, a "right" move on the stairstep path corresponds to an open parenthesis, and an "up" move corresponds to a letter. Then write one extra letter at the end. There will be only one way to insert the close parentheses so the products come in pairs. (The proof is left as an exercise for the enthusiastic reader.)

Exercises

For each problem determine if the given function f represents the Catalan numbers. That is, does $f(n) = C_n$. If not, describe how to modify the function f so that $f(n) = C_n$.

Note: Lower lever students would not be expected to prove their answers. Students can answer the questions by computing the first few values of the given functions. Upper level students should be encouraged to provide proofs by explicitly demonstrating a bijection between the set being counted and any one of the sets which have already been shown to be countable using Catalan numbers.

1. An election will be held between two candidates (candidate A and candidate B) where $2n$ people will vote one after the other. Let $f(n)$ represent the number of ways the $2n$ people can vote such that candidate A is never behind. (Answer, No. The each candidate must finish with n votes.)

2. Let $f(n)$ represent the number of plane rooted trees with n edges. (Answer, Yes)

3. Let $f(n)$ represent number of ways to triangulate a regular n-gon. (Answer, No. The n-gon should be changed to an $(n + 2)$-gon)

4. Let $f(n)$ represent the number of ways $2n$ people, seated around a table can shake hands in n pairs, without their arms crossing.

5. Dick and Jane are playing "rock, paper, scissors." Let $f(n)$ represent the number of ways that Dick and Jane can play $2n$ games and each win n games. (Answer, No. The condition "Dick is never behind" or "Jane is never behind" must also be added. You may wish to discuss the conditions "Dick is always ahead" or "One player is never behind." Note that the second condition would result in $f(n) = 2C_n$.)

6. Let $f(n)$ represent the number of mountain ranges you can draw with n up strokes and n down strokes (no valleys below sea level are allowed).