

# Tropical problems for an abstract algebra course

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## Abstract

In “tropical mathematics” the ordinary operations of addition and multiplication are replaced with  $\min$  and  $+$  respectively. This point of view has interesting applications in control theory, optimization, and phylogenetics and has recently received renewed attention among researchers in combinatorics and algebraic geometry. From the standpoint of abstract algebra, these idempotent semirings provide an unusual and instructive running example. Here we simply suggest a sequence of problems that could be easily integrated into a first course in abstract algebra.

## Introduction and Goals

The tropical semiring has recently received a good deal of attention among researchers in combinatorics and algebraic geometry. The interested reader can find many accessible references by searching for “tropical” at [www.arXiv.org](http://www.arXiv.org). The algebraic operations are quirky, but accessible. This makes them ideally (no pun intended) suited as an example to stretch the thinking of students in an abstract algebra course.

Our goal here is to give a sequence of problems that could be integrated into an abstract algebra course without requiring a great deal of class time. We include problems with a brief statement about where they might appear and teaching points.

## Problems *with commentary*

1. Let  $R = \mathbb{Z}, \mathbb{Q}$ , or  $\mathbb{R}$  with the element  $\infty$  adjoined. Redefine the basic arithmetic operations as follows:

$$x \oplus y := \min\{x, y\} \quad \text{and} \quad x \odot y := x + y.$$

- (a) Check that  $2 \oplus 5 = 2$  and  $2 \odot 5 = 7$ .
- (b) Work out the “addition” and “multiplication” tables for  $\oplus$  and  $\odot$  for  $\{0, 1, \dots, 5\} \times \{0, 1, \dots, 5\}$ .
- (c) Are these operations associative? commutative? Does either have an identity?
- (d) Prove or give a counterexample to the tropical distributive law:

$$x \odot (y \oplus z) = (x \odot y) \oplus (x \odot z)$$

*This exercise would fit most naturally after the ring axioms are introduced. One unusual feature of  $\mathbb{T}\mathbb{Z}$  is that while “multiplicative” inverses are easy, “additive” inverses do not exist in general. The point can also be made that elementary school children would find learning their tropical math facts much easier than the usual ones.*

2. Which of the following make sense in  $\mathbb{T}\mathbb{Z}$ , the tropical integers? Compute those that do.  $1/2, 7/3, 7 - 3, 3 - 7, \sqrt{6}, \sqrt{7}, \sqrt[3]{8}, \sqrt[3]{12}$ ?

*An exercise to reinforce tropical arithmetic and taking inverses.*

3. In the tropical reals  $\mathbb{T}\mathbb{R}$  define,  $x^a := x \odot x \odot \cdots \odot x$  ( $a$  times). Compute  $(x \oplus y)^2$  and  $(x \oplus y)^3$ . Can you generalize to a nice formula for  $(x \oplus y)^n$ ? What does this say about the “tropical Pascal’s triangle”?
4. Consider the “tropical linear equation”  $a \odot x = b$ , where  $a, b, x \in \mathbb{T}\mathbb{R}$ . For what values of  $a$  and  $b$  is there a unique solution  $x$ ? (Try several examples.) Give a complete description of how many solutions this equation has in terms of  $a$  and  $b$ .
5. Consider the “tropical quadratic equation”  $q(x) = x^2 \oplus 2 \odot x \oplus 5$ .
- Graph this equation in the tropical plane.
  - What are the “critical points,” i.e, values of  $x$  where the function changes behavior?
  - Can you factor  $q(x)$  as a product of linear factors? What do you notice?
  - Repeat the above for  $r(x) = 3 \odot x^2 \oplus 2 \odot x \oplus 1$ .
  - Can you generalize this to the generic quadratic equation  $q(x) = a \odot x^2 \oplus b \odot x \oplus c$ ?
6. Consider the generic cubic equation  $p(x) = a \odot x^3 \oplus b \odot x^2 \oplus c \odot x \oplus d$  in  $\mathbb{T}\mathbb{R}[x]$ . When can you factor this into linear factors?
7. Try to generalize your work in the previous problem to come up with a “Tropical Fundamental Theorem of Algebra”

*The above exercises would fit into sections where students are working with polynomials over various rings. It’s interesting that the “bends” in the graphs of the equations really correspond to the “zeroes” of the function. One caveat is that apparently different polynomials can define the same function, just as in  $\text{mod } 5$   $x^5$  and  $x$  define the same function. Modulo this issue, the Tropical FTA is much easier to prove than its counterpart for  $\mathbb{C}[x]$ .*

## References

- [1] D. Speyer and B. Sturmfels. Tropical Mathematics. Preprint.
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