Tropical Arithmetic in Mathematics Courses for Liberal Arts Students or for Elementary Education Majors

Darrell Allgaier (Grove City College)
David Perkinson (Reed College),
Sarah Ann Stewart (North Central College)
John Thurber (Eastern Oregon University)

Abstract

The purpose of this module is to provide an introduction to the system of tropical arithmetic. It is intended for use in courses with students who have little background in mathematics and are unlikely to study further mathematics beyond the course in which they are currently enrolled. It provides a contrast to the standard model of arithmetic with an opportunity to analyze and compare basic axioms and elementary properties of the two models. It also allows an introduction to a topic of current research interest to the professional mathematics community.

Introduction and Goals

Tropical mathematics is the study of properties of the *tropical semiring* $(\mathbb{R}\cup\{\infty\}, \oplus, \otimes)$ with the binary operations defined below. (One need not know the definition of *semiring*, just proceed with the definitions below.) We use this to provide non-mathematics majors with an example of alternative or unusual mathematical systems which are useful to people in fields other than mathematics and which provide objects of continuing interest to professional mathematicians. It also provides an opportunity to enhance the understanding of axiomatic systems and their consequences.

The module includes an introduction to the system $(\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ followed by discussion questions and exercises. The questions and exercises can be easily adapted for use either as individual homework assignments or for group work. One implementation could be to split the module over two days, with the first day's activity including appoximately five minutes of introductory discussion to be followed by a handful of exercises in preparation for the second day's activitiy.

Axioms

The basic arithmetic operations of "addition" and "multiplication" on this system are as follows:

$$x \oplus y := \min(x, y) \qquad \text{and} \qquad x \otimes y := x + y$$

Warm-up Exercises

1. Evaluate the following expressions to familiarize yourself with the operations.

$3 \oplus 7$	$3 \otimes 7$
$0 \oplus 7$	$0 \otimes 7$
$1 \oplus 7$	$1 \otimes 7$
$3 \otimes (7 \oplus 4)$	

Discussion Questions

1. Is it always true that $a \oplus b = b \oplus a$? What about $a \otimes b = b \otimes a$? Is it always true that $a \oplus 0 = a$?
2. In standard arithmetic we have the distributive law which says that for any numbers a, b, c, we have $a \times (b + c) = (a \times b) + (a \times c)$. Does this rule still hold in the new system, that is, is it always true that $a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$?
3. In standard arithmetic, the number 0 functions as an *additive identity*. That is, for any number a we have $0 + a = a + 0 = a$. Is there any number in the new system that serves the same function? (*Note:* This serves as a chance to introduce a new symbol, ∞, to the system to fill this role.)
4. In standard arithmetic, the number 1 functions as a *multiplicative identity*. That is, for any number a we have $1 \times a = a \times 1 = a$. Is there any number in the new system that serves the same function?
5. Try to find examples of operations that work in standard arithmetic but fail in tropical arithmetic, or vice versa.

Tropical graphing (Lines)

In standard geometry we can sketch the graphs of equations in the x-y plane. For example, we know that an equation of the form $y = mx + b$ corresponds to the graph of a line with slope m and y-intercept b.

1. In the standard system, sketch the graph of $y = 2x + 1$. Now sketch the graph of the $y = (2 \otimes x) \oplus 1$.
2. If we consider graphs of equations of the form $y = (m \otimes x) \oplus b$, what happens as we vary the coefficient m?
3. In tropical algebra, is there a standard form for a polynomial whose graph is a line in the plane?
4. What aspect of the tropical equation of a line tells us whether a pair of lines are parallel?

Polynomials

In standard arithmetic, we understand the operation of exponentiation:
$a^2 = a \cdot a, \quad a^3 = a \cdot a \cdot a, \quad \text{etc.}$
In tropical mathematics the natural interpretation is then
$a^2 = a \otimes a, \quad a^3 = a \otimes a \otimes a, \quad \text{etc.}$

1. Sketch the the graphs of the standard $y = x^2$ and the tropical $y = x^2$
2. Compare the graphs of the standard $y = 1 \cdot x^2$ and the tropical $y = 1 \otimes x^2$.
3. In standard arithmetic it is clear, for example, that $(2+5)^2 \neq 2^2 + 5^2$. Thus it is not true in general that $(a+b)^2 = a^2 + b^2$. However, it can be shown that the latter equation IS true in tropical mathematics. Verify this for a few examples. After having verified it for a few examples, try to work through the algebra and show that the equation must hold for all a and b. Remember to make use of the distributive law and the definitions of the operations.

Factoring

1. In standard algebra it can be shown that $x^2 + 4x + 3 = (x+1) \times (x+3)$. This is verified by multiplying out the right hand side of the equation. The analogous claim in tropical algebra would be $x^2 \oplus (4 \otimes x) \oplus 3 = (x \oplus 1) \otimes (x \oplus 3)$. Determine whether this tropical equation is valid, either by sketching graphs of the expressions on either side of the equal sign, or by working through the definitions of the operations for either side.
2. (*Bonus!*) The previous example shows that factoring in our familiar standard way does not translate effectively to the tropical system. It turns out that there are conditions on a, b, and c under which a quadratic tropical polynomial of the form $(a \otimes x^2) \oplus (b \otimes x) \oplus c$ may be factored. Try to detemine these conditions, and demonstrate the factored form of a polynomial which satisfies these conditions.

References

[1] D. Speyer, and B. Sturmfels: Tropical Mathematics, PCMI 2004 handout.