Outreach Lecture: Polytopes

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Abstract

This is an outline of an introduction to polytopes for high school students, culminating in a discussion of the f-vectors of 4-polytopes.

Introduction and Goals

The ultimate goal of this lecture is to let high school students see that mathematics is a living and engaging subject by introducing them to a current research question. We start in Part I by showing that the study of higher-dimensional polytopes is natural and important (e.g. linear programing and the traveling salesman problem) and also beautiful (Schlegel diagrams of 4-polytopes).

The f-vector of a polytope is a list of the number of faces in each dimension. In Part II, we outline a proof of a theorem of Steinitz, characterizing the f-vector of 3-polytopes. On the way, we establish Euler's formula. To keep the high school teachers from being bored at this point, we give an unusual proof involving electrical charges.

In Part III, we reveal that it is an open problem to characterize the f-vectors of 4-polytopes. We say something about what is known and Ziegler's program to describe the f-vectors (cf. Ziegler's lecture notes, 2004 PCMI).

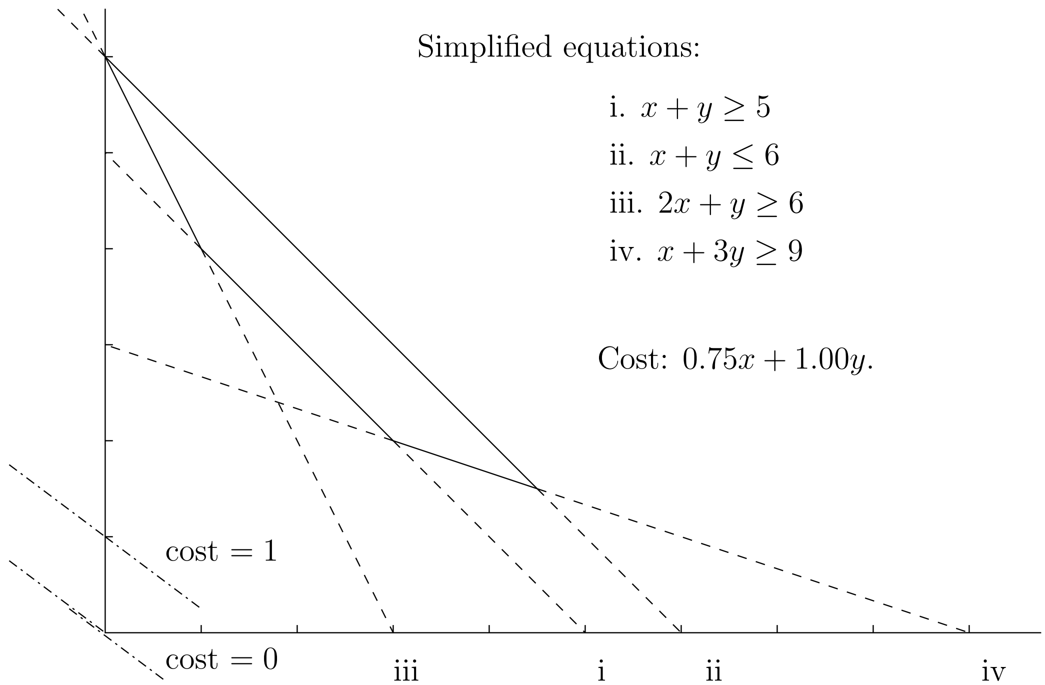
Ideally, the speaker would have access to the program POLYMAKE, especially to visualize the Schlegel diagram of 4-polytopes. We have included a collection of overheads and POLYMAKE files that could be used for this talk.

Part I: Examples.

- **Introduction.**
  - Start with examples of polytopes in 1, 2, and 3 dimensions. Nice pictures: the Platonic solids, the Archimedean solids, a random polytope with 1000 vertices, good picture illustrating duality. Bring models.
  - The introduction leads to the definition of a polytope as the convex hull of a finite set of points or as the bounded intersection of halfspaces. Examples of closely related objects: stellated polyhedra, flexahedra.
- **Linear programming example.**

– Two-dimensional example from linear programming. We have two hypothetical brands, x and y, with the listed number of calories, units of vitamins A and C, and costs. We require between 300 and 360 calories, at least 36 units of vitamin A, and at least 90 units of vitamin C. Asking to minimize the cost leads to a linear programming problem whose solution is given by the southwest corner of the a certain quadrilateral.

Brand	Cal	A	C	Cost
x	60	12	10	\$ 0.75
y	60	6	30	\$ 1.00



Simplified equations:

i. $x + y \geq 5$

ii. $x + y \leq 6$

iii. $2x + y \geq 6$

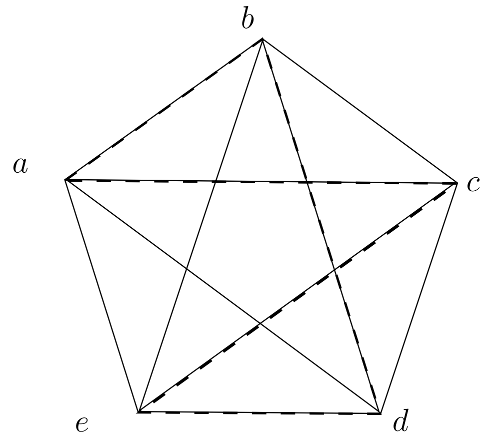
iv. $x + 3y \geq 9$

Cost: $0.75x + 1.00y$.

What happens in a linear programming problem if there are more variables? This question leads to a discussion of higher-dimensional polytopes.

– Traveling salesman polytope.

Consider travel between cities a, b, c, d, and e.



Each tour that passes through each city and uses no path twice corresponds to a point in 10-dimensional space having only 0s and 1s as coordinates. For example, the path illustrated above is given by the point $(1, 1, 0, 0, 0, 1, 0, 0, 1, 1)$:

edge:	ab	ac	ad	ae	bc	bd	be	cd	ce	de
	1	1	0	0	0	1	0	0	1	1

Each coordinate corresponds to a path between two cities, a 1 signifying that the path is used in the tour.

It turns out there are 12 tours between our five cities, giving 12 points in $\mathbb{R}^{10}$. The convex hull of this set of points is a polytope of dimension 5 with vertices corresponding to the 12 tours called the traveling salesman polytope, TSP(5). (The f-vector is $(12, 60, 120, 90, 20)$.)

Suppose we assign a cost to each path:

edge:	ab	ac	ad	ae	bc	bd	be	cd	ce	de
\$ cost:	1	1	10	5	3	1	2	6	1	1

In $\mathbb{R}^{10}$, if we move in the direction opposite the vector $(1, 1, 10, 5, 3, 1, 2, 6, 1, 1)$, the extreme vertex of TSP(5) is the vertex corresponding to the path pictured above. [$\dim \mathrm{TSP}(n) = n(n-3)/2$ and the number of vertices of $\mathrm{TSP}(n)$ is $(n-1)!/2$.]

- **Schlegel diagrams of 4-polytopes.**
  - Start with a 2-dimensional Schlegel diagram of a cube and a 1-dimensional Schlegel diagram of a square, then give the 3-dimensional Schlegel diagram of a 4-cube.
  - More examples: 4-simplex, product of n-gons, the 120-cell.

Part II: Euler's formula and the f-vectors of 3-polytopes.

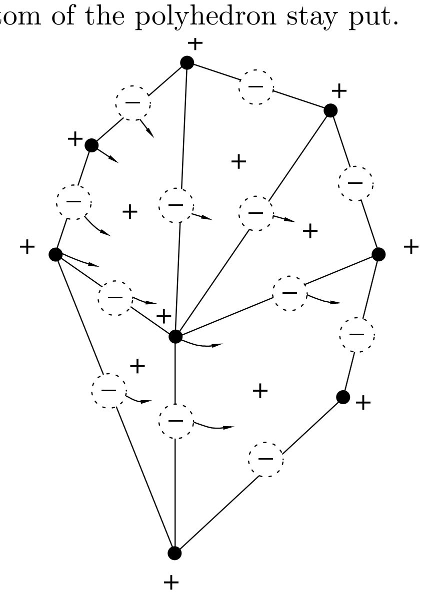
- **Introduction.** What is an f-vector? Examples/pictures/duality. What are the possible f-vectors of 2-polytopes?

- **Euler's formula.** $f_0 - f_1 + f_2 = 2$, i.e., $V - E + F = 2$.
  - Formula first appears in a 1750 letter from Euler to Goldbach. The proof given in the letter had a subtle error. The first rigorous proof was given by Legendre in 1794. His proof involved projecting the polyhedron to a sphere, then using metrical properties of the sphere. This proof is rigorous but seems to introduce extraneous ideas (measurements on the sphere).

    Optional. Give Cauchy's 1813 proof. This is the one you usually see: (i) remove a face of P and stretch out the edges to make a planar graph; (ii) by adding edges, you may assume that all the faces are triangles (adding an edge also adds a face, so $V - E + F$ does not change); (iii) deconstruct the graph one vertex at a time—noticing that $V - E + F$ does not change—until you end up with a triangle, for which the theorem is obvious.

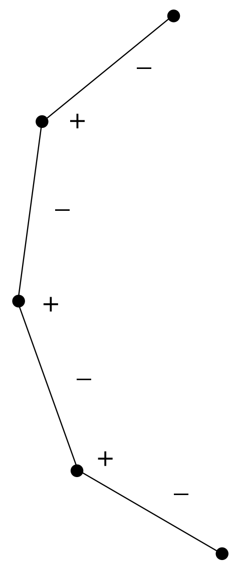
    Why is Cauchy's approach better than Legendre's?
  - To see 17 different proofs of Euler's formula, see:

    www.ics.uci.edu/∼eppstein/junkyard/euler/all.html.
  - Here is a recent "shocking" proof due to Thurston.
    1. Suspend your polyhedron so that there is a unique highest vertex and a unique lowest vertex and no edge parallel to the ground. Further, by wiggling the polyhedron a bit, make sure that each face has a unique highest vertex and a unique lowest vertex.
    2. Put a positive charge on each vertex, a negative charge on each edge, and a positive charge in the middle of each face.
    3. Now let each charge on a vertex or an edge move in a plane parallel to the ground, counter-clockwise. The charges on the faces and those at the vertices at the top on the bottom of the polyhedron stay put.



Front view of an electrically charged polyhedron.

4. How many positive and negative charges have migrated into each face? Looking at a face straight on, only those charges that are on the vertices and edges along the left-hand side move into the face.



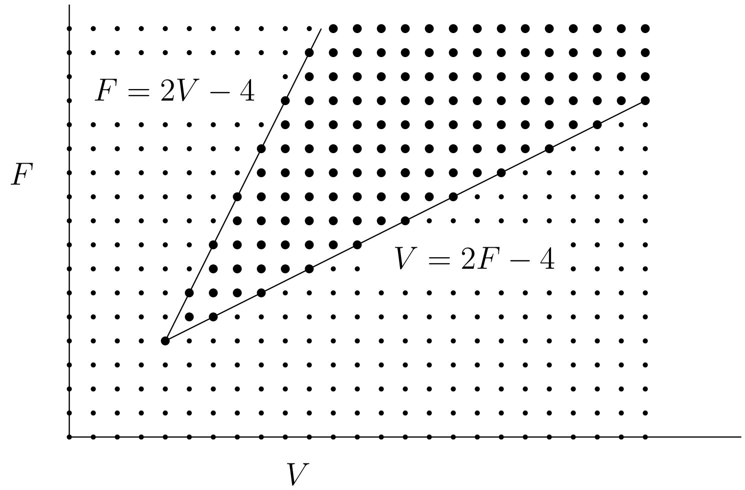
Charges along the left side of a face.

5. Note that the charges at the top and bottom vertex on the group of edges along the left-hand side of the face do *not* move into the face. Counting up the charges, starting from the bottom, you'll see that the charges alternate in sign. In the end, one more negative charge than positive charge moves into the face. Don't forget that there is already a positive charge waiting in this face. Thus, all the charges cancel.
6. All, that is, but the charges sitting at the top and bottom vertices of the polyhedron. Thus, $V - E + F = 2$.

- ***f*-vectors of 3-polytopes.**
  - $2E \geq 3F$ and $2E \geq 3V$. Why? When do you get equality?
  - Substitute $E = V - F - 2$ (Euler's formula) to get

$$F \leq 2V - 4, \qquad V \leq 2F - 4.$$

  When do you get equality?
  - Plot the region determined by these inequalities.



- Each dot in the region corresponds to a polyhedron. Can you identify some?
- Constructions: prisms over n-gons, stacking (adding a vertex near the center of a triangular face), cutting off a vertex. How does this effect V and F? Can you think of a construction that adds one vertex and one face?
- Steinitz's theorem from 1906: Every (V, F) in the region described above is realizable by a polyhedron.
- How do you account for the symmetry of the region?

Part III: f-vectors of 4-polytopes.

- A 4-polytopes has vertices, edges, ridges (2-dimensional), and facets (3-dimensional). The number of each of these is recorded in the f-vector for the polytope: $f = (f_0, f_1, d_2, f_3)$. Examples:
  - 4-dimensional simplex (convex hull of five generic points in $\mathbb{R}^4$): $f = (5, 10, 10, 5)$.
  - 4-cube: $f = (16, 32, 24, 8)$.
- Unlike the case of three dimensions, the f-vectors of 4-polytopes have not been completely described. Known constraints:
  - $f_0 \geq 5, \qquad f_3 \geq 5, \qquad f_1 \geq 2f_0, \qquad f_2 \geq 2f_3.$
  - $f_0 - f_1 + f_2 - f_3 = 0$, (Euler's formula).
  - $2f_1 + 2f_2 \geq 5f_0 + 5f_3 - 10$ (not obvious)
- The exact pairs (f_0, f_3) arising from 4-polytopes is known. They are exactly the integers satisfying

$$\begin{aligned} 5 \leq f_0 &\leq \frac{1}{2} f_3(f_3 - 3) \\ 5 \leq f_3 &\leq \frac{1}{2} f_0(f_0 - 3) \end{aligned}$$

(Plot this region.) [The other pairs are also known. Reference: Grünbaum's *Convex Polytopes*, second edition, pp. 191–197.]

- Ziegler's program to describe the f-vectors of 4-polytopes (in some sense) by 2006. See the notes for Ziegler's 2004 PCMI lecture.