

# Geometric Combinatorics

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Geometric combinatorics refers to a growing body of mathematics concerned with counting properties of geometric objects described by a finite set of building blocks. Primary examples include polytopes (which are bounded polyhedra and the convex hulls of finite sets of points) and complexes built up from them. Other examples include arrangements and intersections of convex sets and other geometric objects. As we'll see there are interesting connections to linear algebra, discrete mathematics, analysis, and topology, and there are many exciting applications to economics, game theory, and biology.

There are many topics that could be discussed in a course on geometric combinatorics; but in these lectures I have chosen what I consider to be my favorite topics for inclusion in such a course. Mostly the lectures reflect what I like in the subject. Some of the topics are of relatively recent development and reflect either my current interests (e.g., combinatorial fixed point theorems) or material I assimilated in the "Discrete and Computational Geometry" program at MSRI in Fall 2003.

## Workshop Goals

- To introduce one to a selection of topics in geometric combinatorics
- To equip one to begin reading literature in the area
- To enrich one's teaching with new examples or new course ideas
- To stimulate ideas for research problems for oneself or one's students

## Format

There will be a mixture of lectures and group work. The latter will be designed to encourage one to assimilate the material by thinking about problems, or discussing how one might incorporate some of this material in one's work with students. Groups will make presentations on Friday.

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## Tentative Schedule

### — Sunday —

- 9am : Welcome
  - 9:30 : Lecture 1 – Combinatorial Convexity
  - 10:30 : BREAK
  - 11:00 : Lecture 2 – Set Intersections and Helly’s theorem
  - noon : LUNCH
  - 1:15pm : Lecture 3 – Polytopes I: Examples and Construction
  - 2:00 : Group Work
  - 2:45 : BREAK
  - 3:15 : Lecture 4 – Polytopes II: Polar Duality
  - 4:00 : Group Work
  - 5:00 : Adjourn
- 6:00pm : Optional Dinner at Local Restaurant (to be arranged)

### — Monday —

- 9am : Cross-Program Lecture– WeBWorK
  - 10:00 : BREAK
  - 10:30 : Group Work
  - 11:00 : Lecture 5 - Polytopes III: Combinatorics of Faces
  - noon : LUNCH
  - 1:15pm : Lecture 6 - Polytopes IV: Counting Faces
  - 2:15 : Group Work
  - 2:45 : BREAK
  - 3:15 : Lecture 7 - Simplicial Complexes and Triangulations
  - 4:15 : Group Work
  - 5:00 : Adjourn
- 6:00pm : Optional Dinner at Local Restaurant (to be arranged)

### — Tuesday —

- 9am : Lecture 8 - Combinatorial Fixed Point Theorems I : Sperner’s Lemma
- 10:00 : BREAK
- 10:30 : Group Work
- 11:00 : Lecture 9 - Combinatorial Fixed Point Theorems II : Tucker’s Lemma
- noon : LUNCH
- 1:15pm : Lecture 10 - Combinatorial Fixed Point Theorems III : Kneser Colorings
- 2:15 : Adjourn for Free Time (optional excursion to SF)

**— Wednesday —**

- 9am : Lecture 11 - Combinatorial Fixed Point Theorems IV: Trees
  - 10:00 : BREAK
  - 10:30 : Group Work
  - 11:00 : Lecture 12 - An Introduction to Phylogenetic Trees
  - noon : LUNCH
  - 1:15pm : Discussion – Making Undergraduate Research Work
  - 2:15 : Group Work
  - 2:45 : BREAK
  - 3:15 : Group Work
  - 4:00 : Lecture 13 - What is Tropical Geometry?
  - 5:00 : Adjourn
- 7:00pm : Optional Dinner at Local Restaurant (to be arranged)

**— Thursday —**

- 9am : Lecture 14 - Minkowski's Theorem
- 10:00 : BREAK
- 10:30 : Group Work
- 11:00 : Lecture 15 - What are Ehrhart Polynomials?
- noon : LUNCH
- 1:30pm : Group Presentations
- 2:45 : BREAK
- 3:15 : Group Presentations
- 4:15 : Wrap-Up
- 5:00pm : Adjourn

## Some Open Problems

1. Throw  $k$  points down in the unit square and find the area of the largest convex set in the square containing none of the  $k$  points. Let  $f(k)$  be the minimum (of the largest areas) over all sets of  $k$  points. Find good upper and lower bounds on  $f(k)$ .

(a problem of Moser, as told by Lay [4, p.92] For  $k = 3$  it is known that  $1/3 \leq f(3) \leq \sqrt{2}/4$ .)

2. What is the largest area that a  $n$ -gon of unit diameter can have?

(A problem discussed in [4, p.92]. For odd  $n$ , the polygon must be regular. For  $n = 4$  there are infinitely many quadrilaterals. For  $n = 6$ , Graham [3] shows that there is a non-regular hexagon that achieves maximum area, and it is unique.)

3. A Radon relative: can every set of 8 points in the plane be partitioned to form 2 triangles and a line segment so that the segment cuts the interior of both triangles?

(A problem discussed in [1, p.137], related to Tverberg's generalization of Radon's lemma)

4. Let  $\mathcal{A}$  be a finite family of translates of a convex set in  $\mathbb{R}^2$ . Prove or disprove that if every two members of  $\mathcal{A}$  intersect, then some set of 3 points intersects every member of  $\mathcal{A}$ .

(from the Wenger article in [2, p.71]. There are many more Helly-type open problems mentioned in the Wenger article.)

5. Let  $P$  be a  $d$ -polytope, and let  $f_i$  be the number of faces of dimension  $i$ . Prove that  $f_i \geq \min\{f_0, f_{d-1}\}$ .

(a problem of I. Barany, as told by G. Kalai in a Fall 2003 MSRI workshop)

6. Is the graph of triangulations of every 2-dimensional point set  $(n - 3)$ -connected?

(from Aug 2003 MSRI workshop on triangulations, DeLoera et al. It is known all triangulations have at least  $n - 3$  flips. )

7. Let  $A$  be a set of  $n$  points in the plane, and consider all triangulations of  $A$  that use all the vertices. Find a good upper bound of the form  $C^n$ .

(from Aug 2003 MSRI workshop on triangulations, DeLoera et al. The best known upper bound is  $59^n$ . See [5]. It is believed that there are at most  $8^n$  of them. )

8. The Perron-Frobenius theorem says that if  $A$  is a non-negative  $n \times n$  matrix, then  $A$  has a non-negative eigenvalue. One proof of this theorem uses the Brouwer fixed point theorem. Find a proof of this theorem that directly uses Sperner's lemma.

9. Chamber complexes of polytopes are not well understood at all. Study the chamber complex of any polytope and try to explain its structure. For instance, what about the prism over a regular triangle, or any prism over a regular polygon? Can you enumerate the number of chambers?

10. A generalization of the Borsuk-Ulam theorem due to Yang [6, 7] says: for every continuous function  $S^{dn} \rightarrow \mathbb{R}^d$ , there exist  $n$  mutually orthogonal diameters whose  $2n$  endpoints are mapped to the same point. A theorem of Kakutani-Yamabe-Yujobo says: for every continuous function  $S^n \rightarrow \mathbb{R}$ , there exist  $n + 1$  mutually orthogonal radii whose  $n + 1$  endpoints are mapped to the same point. Find a combinatorial analogue for either of these theorems. (Start with small  $n$  and  $d$ .)

## References

- [1] Hallard T. Croft, Kenneth J. Falconer, and Richard K. Guy. *Unsolved problems in geometry*. Springer-Verlag, New York, 1991.
- [2] Jacob E. Goodman and Joseph O'Rourke, editors. *Handbook of discrete and computational geometry*. CRC Press Series on Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL, 1997.
- [3] R. L. Graham. The largest small hexagon. *J. Combinatorial Theory Ser. A*, 18:165–170, 1975.
- [4] Steven R. Lay. *Convex sets and their applications*. Robert E. Krieger Publishing Co. Inc., Malabar, FL, 1992. Revised reprint of the 1982 original.
- [5] Francisco Santos and Raimund Seidel. A better upper bound on the number of triangulations of a planar point set. *J. Combin. Theory Ser. A*, 102(1):186–193, 2003.
- [6] Chung-Tao Yang. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. I. *Ann. of Math. (2)*, 60:262–282, 1954.
- [7] Chung-Tao Yang. On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. II. *Ann. of Math. (2)*, 62:271–283, 1955.

## (My) Principles for Undergraduate Research

### My philosophy about undergraduate research:

- Undergraduates can do research, and often, publishable research.
- For the undergraduate, the process is more important than the outcome.
- My role as advisor is not to be the expert in all things, but to show the student how to become an expert in one thing.
- If one understands a problem well enough, eventually one will see something new.
- If one tries to prove a theorem without looking at the original proof, one will often find a new proof.
- Clear thinking is equivalent to clear writing.

### About choosing problems:

- Pick problems with small, do-able steps.
- Modify a known problem by changing the hypothesis, changing the proof method, or changing the context.
- Try to make connections between areas that haven't been connected before.

### About helping students succeed:

- Build confidence by giving the student do-able goals for each meeting. (And then expect them to meet those goals.)
- Give the student some ownership of the research direction. What questions about the given problem does she find interesting?
- When reading related work, encourage students to think before they read— to discover ideas for themselves before seeing how others have done it.
- Encourage the student to always be writing up (LaTeX) their work. Require a written final report at the end of the project period. The last 2 weeks should be devoted exclusively to writing.
- Give constant reassurance.
- Let the students do the talking. They'll learn how to explain their ideas to others. Encourage them to talk to each other.