THREE PROBLEM SOLUTIONS

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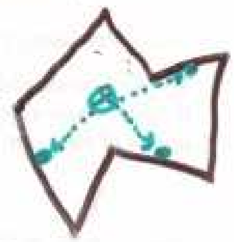
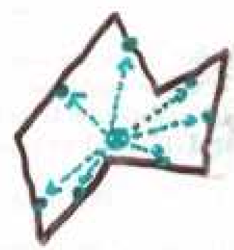
2. ART GALLERY: A POLYGONAL ROOM WITH ONE PAINTING ON EACH WALL. FOR ANY 3 PAINTINGS, $\exists$ POINT IN GALLERY FROM WHICH YOU CAN SEE ALL THREE.

MUST THERE BE A POINT FROM WHICH ALL PAINTINGS ARE VISIBLE?

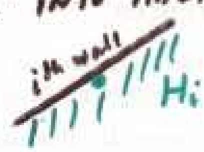




SOLUTION: YES.

EACH WALL, EXTENDED, DIVIDES $\mathbb{R}^2$ INTO HALF-PLANES

LET H_i = "INSIDE" HALF PLANE OF i^{th} wall.

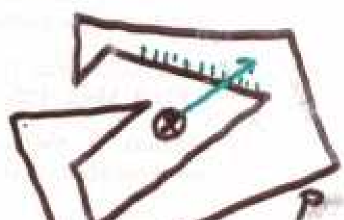


① POINT FROM WHICH WE CAN SEE i, j, k MUST BE IN $H_i \cap H_j \cap H_k$, SO ANY THREE HAVE COMMON POINT

② BY HELLY, $\exists x \in \bigcap H_i$

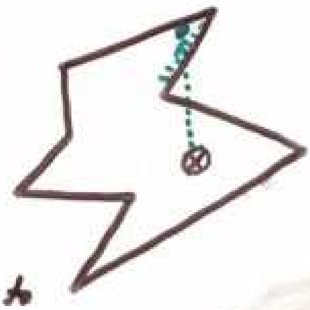
③ x must be in P.

If not, draw ray from x which intersects P. When crosses into P we find i s.t. $x \notin H_i$

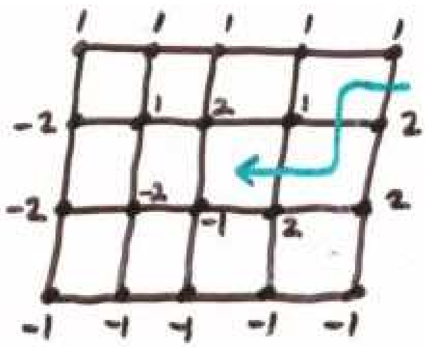


(4) Every painting is visible from x.

If ray from x to some painting crosses a wall, first crossing ~~takes~~ it outside P. To get to painting it must reenter P thru a wall i s.t. $x \notin H_i$



8.



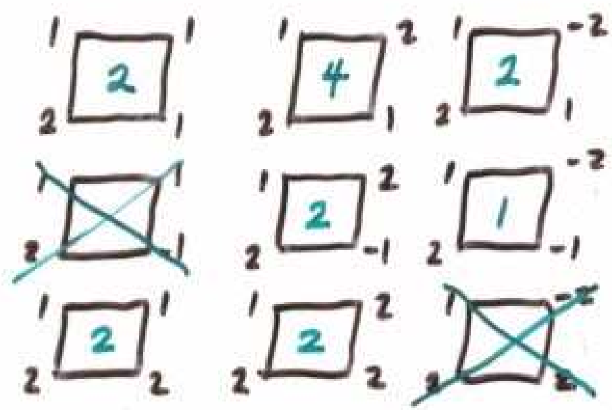
SHOW MUST BE

SOLUTION: ASSUME NO

(1) COUNT 12 edges in squares

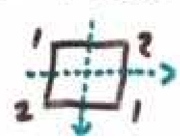
- Each interior 12 edge appears in two squares, and ∃ exactly one boundary 12 edge, so total is odd.

- Possible squares with 12 edges:



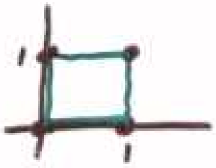
SO MUST BE ODD # OF COMPLETE SQUARES

(2) FOLLOW 12 "doors". (



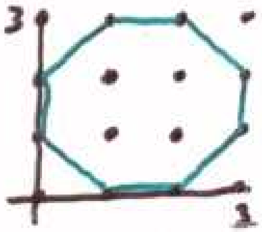
)

9. SUPPOSE CONVEX n-gon in <u>first quadrant</u> has vertices at <u>lattice points</u>. $D(n)$ = smallest <u>maximal</u> <u>coordinate</u>. How fast does $D(n)$ grow?



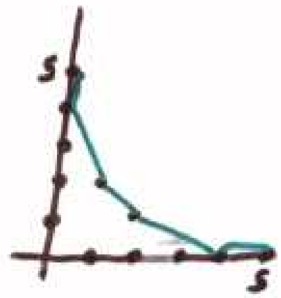
$D(4) = 1$

$D(n)$ is non-decreasing



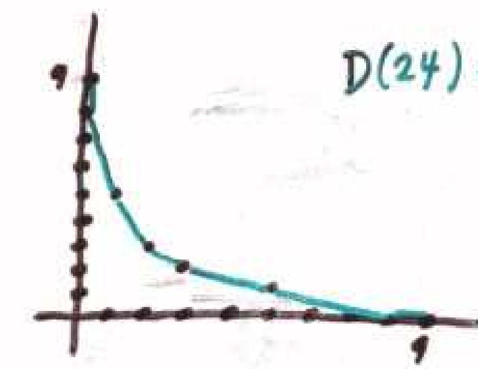
$D(8) = 3$

Add slopes $\frac{1}{2}$, $\frac{2}{1}$



$D(16) = 9$

Add slopes $\frac{1}{3}$, $\frac{3}{1}$



$D(24) = 17$

Add slopes
$\frac{1}{4}$, $\frac{2}{3}$, $\frac{3}{2}$, $\frac{4}{1}$

$D(40) = 37$

	k	a_k	$b_k = D(a_k)$
	1	4	1
	2	8	3
	3	16	9
	4	24	17
	5	40	37
add $\frac{1}{5}, \frac{5}{1}$	6	48	49
add $\frac{1}{6}\ \frac{2}{5}\ \frac{3}{4}\ \frac{4}{3}\ \frac{5}{2}\ \frac{6}{1}$	7	72	91
	$\vdots$	$\vdots$	$\vdots$

At k^{th} stage, add all fractions $\frac{a}{b}$ with $a+b=k$

$\gcd(a,b)=1$

i.e. $\gcd(a,k)=1$

Augment a_k by $4\phi(k)$

b_k by $k\phi(k)$

$\phi(k)$ = # of positive integers $< k$ and relatively prime to k

$(\phi(1)=1)$

so $a_k = 4\sum_{i=1}^{k}\phi(i)$

$b_k = \sum_{i=1}^{k} i\phi(i)$

ONLINE HANDBOOK OF INTEGER SEQUENCES

NEIL SLOANE

www.research.att.com/~njas/sequences

$a_k \sim \frac{3k^2}{\pi^2}$ $\qquad b_k \sim \frac{2}{\pi^2} k^3$

$D(4a_k) = b_k$

$$\boxed{D(n) \sim \frac{\pi}{12\sqrt{3}} n^{3/2} \approx .15115\, n^{3/2}}$$

n	$D(n)$	$.15\, n^{3/2}$
8	3	3.4
16	9	9.6
24	17	17.6
40	37	37.9
48	49	49.9
72	91	91.6
88	123	123.8
⋮	⋮	⋮