Discrete dynamical systems terminology

Assume throughout that (S, f) is a given discrete dynamical system (where S is a state space, $f : S \to S$ is function on that state space).

fixed point

A state $x \in S$ is a *fixed point* of f if $f(x) = x$.

periodic point

For a positive integer k, $x \in S$ is a *periodic point* of f (of period k) if $f^k(x) = x$.

prime period

Given a periodic state x, if k is the smallest positive integer k such that $f^k(x) = x$, then k is the *prime period* of x.

period-k orbit

If $p_1, p_2, \ldots, p_k \in S$ are such that

$$f(p_1) = p_2, f(p_2) = f(p_3), \ldots, f(p_k) = p_1,$$

then $\{p_1, p_2, \ldots, p_k\}$ is a *period-k orbit* of f.

The Multiplier Thm
(stability of fixed points)

Suppose p is a fixed point of f where f is a "smooth" function (i.e. differentiable and f' continuous). Then

- $|f'(p)| < 1 \Longrightarrow p$ is attracting (stable),
- $|f'(p)| > 1 \Longrightarrow p$ is repelling (unstable),
- $|f'(p)| = 1 \Longrightarrow$ test is inconclusive (could be stable or unstable or neither).

The Multiplier Thm for periodic orbits
(stability of periodic orbits)

Let f be a "smooth" function. Suppose $\{p_1, p_2 \ldots, p_k\}$ is a period-k orbit of f. Then

- $|f'(p_1)f'(p_2)\cdots f'(p_k)| < 1 \Rightarrow$ periodic orbit is attracting,
- $|f'(p_1)f'(p_2)\cdots f'(p_k)| > 1 \Rightarrow$ periodic orbit is repelling,
- $|f'(p_1)f'(p_2)\cdots f'(p_k)| = 1 \Rightarrow$ test is inconclusive.

bifurcation

A *bifurcation* is an abrupt change in the qualitative behavior of a parameterized family of dynamical systems as a parameter is varied.

bifurcation diagram

A *bifurcation diagram* shows the possible long-term values of a family of dynamical systems as a function of a parameter in the systems.

Sarkovskii ordering

Ordering of the positive integers as follows:

$$\begin{array}{cccccc} 3 & 5 & 7 & 9 & 11 & \cdots \\ 2\cdot 3 & 2\cdot 5 & 2\cdot 7 & 2\cdot 9 & 2\cdot 11 & \cdots \\ 2^2\cdot 3 & 2^2\cdot 5 & 2^2\cdot 7 & 2^2\cdot 9 & 2^2\cdot 11 & \cdots \\ & & \vdots & & & \\ \ldots & 2^n & \cdots & 2^2 & 2^1 & 1 \end{array}$$

Sarkovskii's Thm

Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and that f has a periodic point of prime period n. If n precedes m in the Sarkovskii ordering, then f also has a periodic point of prime period m.

Corollary to Sarkovskii's Thm

A continuous function $f : \mathbb{R} \to \mathbb{R}$ that has a period 3 point has periodic points of prime period m for all positive integers m.

Chaos

A discrete dynamical system (S, f) is chaotic if

(1) f has **sensitive dependence on initial conditions** or **SDIC** (points that start close need not stay close)
(2) f is **topologically transitive** (the points are mixed up really well)
(3) the periodic points of f are **dense** in S (every point is arbitrarily close to a periodic point)