

Mirror Symmetry Through Reflexive Polytopes

Physical and Mathematical Dualities

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April 2010



Outline

String Theory and Mirror Symmetry

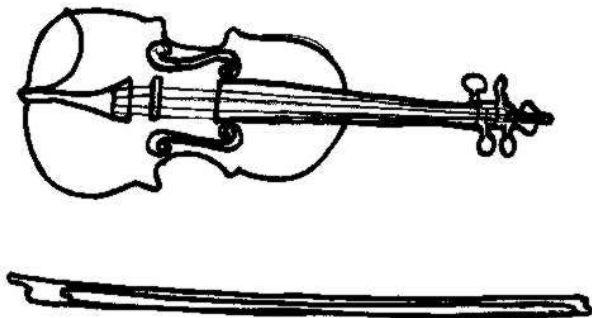
Some Complex Geometry

Reflexive Polytopes

From Polytopes to Spaces

Are Strings the Answer?

- ▶ “Fundamental” particles are **strings** vibrating at different frequencies.



- ▶ Strings **wrap** other dimensions!

T-Duality

Pairs of Universes

An extra dimension shaped like a circle of radius R and an extra dimension shaped like a circle of radius α'/R yield indistinguishable physics! (The slope parameter α' has units of length squared.)

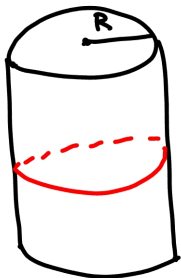


Figure: Large radius, few windings



Figure: Small radius, many windings

Heterotic String Theory

Supersymmetries

Interchange bosons (spin n) with fermions (spin $n + \frac{1}{2}$)

Hybrid Vigor

- ▶ Equations of motion split into “left-moving” and “right-moving” solutions.
- ▶ Operators Q and \bar{Q} act on left-moving and right-moving states, respectively.
- ▶ Q and \bar{Q} are well-defined up to sign.

Building a Model

Locally, space-time should look like

$$M_{3,1} \times V.$$

- ▶ $M_{3,1}$ is four-dimensional space-time
- ▶ V is a d -dimensional complex manifold
- ▶ Physicists require $d = 3$ (6 real dimensions)
- ▶ V is a Calabi-Yau manifold
- ▶ For differential geometers, V is a Ricci-flat Kähler-Einstein manifold
- ▶ Our choice of geometric model fixes a choice of sign for the operators Q and \bar{Q}

Geometry Appears

Physical properties of our model for the universe should correspond to geometric properties of the space V .

We will be particularly interested in the **complex moduli** and **Kähler moduli** of V .

The Heterotic Duality

- ▶ Changing the sign of the operator Q corresponds to choosing a new Calabi-Yau manifold V° .
- ▶ The **geometrical** properties of V and V° yield equivalent **physical** theories.

Mirror symmetry is the study of the mathematical consequences of this duality.

Realizing Mirror Symmetry Geometrically

We need:

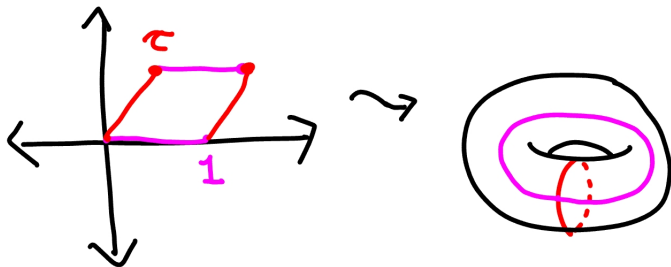
- ▶ Complex manifolds
- ▶ which are Calabi-Yau
- ▶ and arise in paired or “mirror” families
- ▶ with dual geometric properties.

Varying **complex** structure in one family should correspond to varying **Kähler** structure in the other family.

Complex Structure

An n -dimensional **complex manifold** is a geometric space which looks locally like \mathbb{C}^n .

Example: Elliptic Curves



We can think of varying the parameter τ as either changing the complex manifold, or changing the **complex structure** on an underlying topological 2-torus.

Kähler Structure

Standard Product

We can pair vectors v and w with their tails at a point z in \mathbb{C} using the product for complex numbers:

$$\langle v, w \rangle = v \bar{w}$$

Note that $\langle v, v \rangle = \|v\|^2$.

More generally, if $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$ are vectors with their tails at a point $\mathbf{z} = (z_1, \dots, z_n)$ in \mathbb{C}^n , their **standard Hermitian product** is given by

$$\begin{aligned} \langle \vec{v}, \vec{w} \rangle &= \sum v_i \bar{w}_i \\ &= \vec{v}^T \bar{\vec{w}} \end{aligned}$$

Kähler Structure

Hermitian metrics

A **Hermitian metric** H tells us how to pair tangent vectors at any point of a complex manifold and obtain a complex number.

$$H(\vec{v}, \vec{w}) = \overline{H(\vec{w}, \vec{v})}$$

Kähler Structure

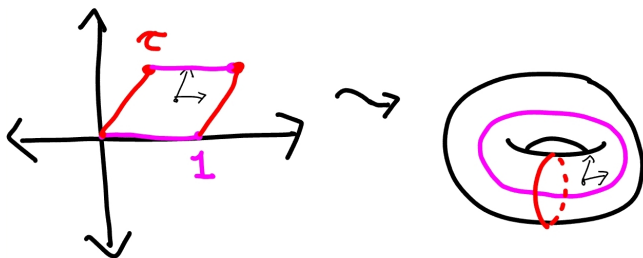
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Elliptic Curve Example

We can use the standard Hermitian product on \mathbb{C} to describe a Hermitian metric for tangent vectors to an elliptic curve.



Kähler Structure

Kähler Metrics

A **Kähler metric** is a special type of Hermitian metric which can be written in local coordinates as follows. If \vec{v} and \vec{w} are vectors with their tails at a point \mathbf{z}_0 in \mathbb{C}^n ,

$$\kappa(\vec{v}, \vec{w}) = \vec{v}^T (\mathbf{I} + \mathbf{G}(\mathbf{z})) \overline{\vec{w}}.$$

Here \mathbf{I} is the identity matrix and

$$\mathbf{G}(\mathbf{z}) = \begin{pmatrix} g_{11}(\mathbf{z}) & \dots & g_{1n}(\mathbf{z}) \\ \vdots & \ddots & \vdots \\ g_{n1}(\mathbf{z}) & \dots & g_{nn}(\mathbf{z}) \end{pmatrix}$$

vanishes up to order 2 at \mathbf{z}_0 .

The Geometric Ingredients of Mirror Symmetry

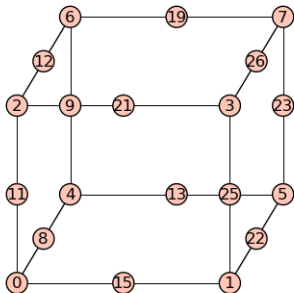
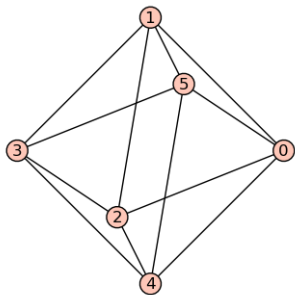
We need:

- ▶ Complex manifolds V
- ▶ which are Calabi-Yau
- ▶ and arise in paired or “mirror” families
- ▶ with dual geometric properties.

Varying **complex** structure in one family should correspond to varying **Kähler** structure in the other family.

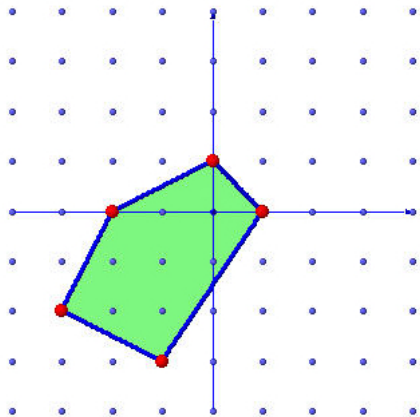
Batyrev's Insight

We can describe mirror families of Calabi-Yau manifolds using combinatorial objects called **reflexive polytopes**.



Lattice Polygons

The points in the plane with integer coordinates form a **lattice** M .
A **lattice polygon** is a polygon in the plane which has vertices in the lattice.



Reflexive Polygons

We say a lattice polygon is **reflexive** if it has only one lattice point, the origin, in its interior.

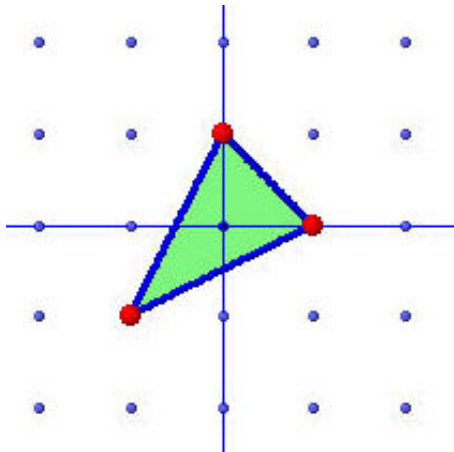
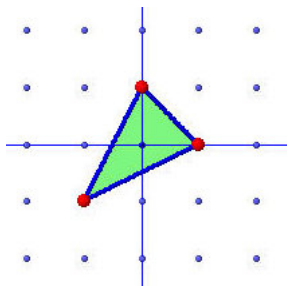


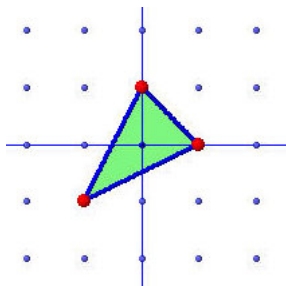
Figure: A reflexive triangle

Describing a Reflexive Polygon



- ▶ List the vertices

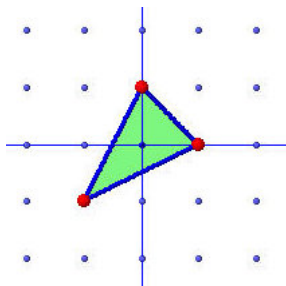
Describing a Reflexive Polygon



- List the vertices

$$\{(0, 1), (1, 0), (-1, -1)\}$$

Describing a Reflexive Polygon

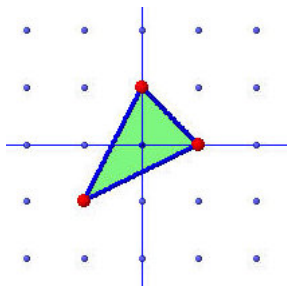


- ▶ List the vertices

$$\{(0, 1), (1, 0), (-1, -1)\}$$

- ▶ List the equations of the edges

Describing a Reflexive Polygon



- ▶ List the vertices

$$\{(0, 1), (1, 0), (-1, -1)\}$$

- ▶ List the equations of the edges

$$-x - y = -1$$

$$2x - y = -1$$

$$-x + 2y = -1$$

A Dual Lattice

Let M be another copy of the points in the plane with integer coordinates.

The dot product lets us pair points in N with points in M :

$$(n_1, n_2) \cdot (m_1, m_2) = n_1 m_1 + n_2 m_2$$

Polar Polygons

Edge equations define new polygons

Let Δ be a lattice polygon in N which contains $(0,0)$. The **polar polygon** Δ° is the polygon in M given by:

$$\{(m_1, m_2) : (n_1, n_2) \cdot (m_1, m_2) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

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$$(x, y) \cdot (-1, -1) = -1$$

$$(x, y) \cdot (2, -1) = -1$$

$$(x, y) \cdot (-1, 2) = -1$$

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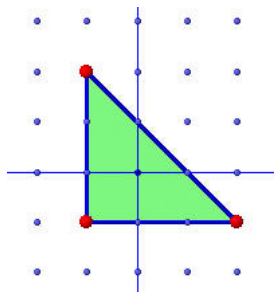


Figure: Our triangle's polar polygon

Mirror Pairs

If Δ is a reflexive polygon, then:

- ▶ Δ° is also a reflexive polygon
- ▶ $(\Delta^\circ)^\circ = \Delta$.

Δ and Δ° are a **mirror pair**.

A Polygon Duality

Mirror pair of triangles

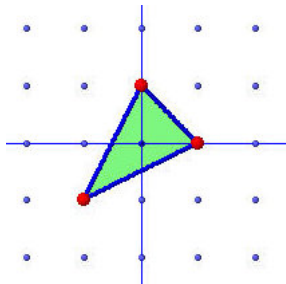


Figure: 3 boundary lattice points

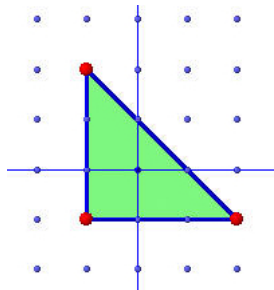


Figure: 9 boundary lattice points

$$3 + 9 = 12$$

Mirror Pairs of Polygons

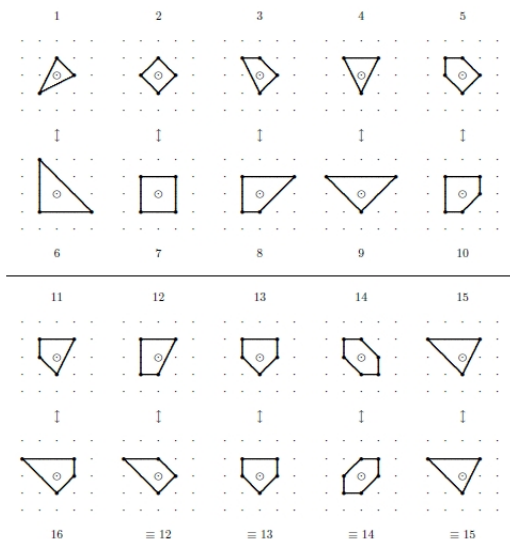
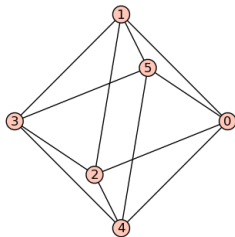


Figure: F. Rohnsiepe, “Elliptic Toric K3 Surfaces and Gauge Algebras”

Other Dimensions

Definition

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_q\}$ be a set of points in \mathbb{R}^k . The **polytope** with vertices $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_q\}$ is the convex hull of these points.



Polar Polytopes

Let N be the lattice of points with integer coordinates in \mathbb{R}^k . A **lattice polytope** has vertices in N .

As before, we have a **dual lattice** M and a dot product

$$(n_1, \dots, n_k) \cdot (m_1, \dots, m_k) = n_1 m_1 + \dots + n_k m_k$$

Definition

Let Δ be a lattice polygon in N which contains $(0,0)$. The **polar polytope** Δ° is the polytope in M given by:

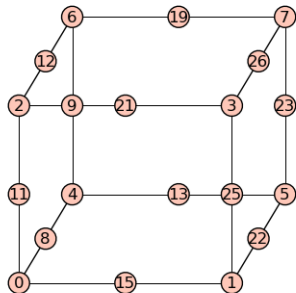
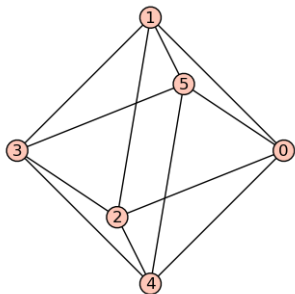
$$\{(m_1, \dots, m_k) : (n_1, \dots, n_k) \cdot (m_1, \dots, m_k) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

Reflexive Polytopes

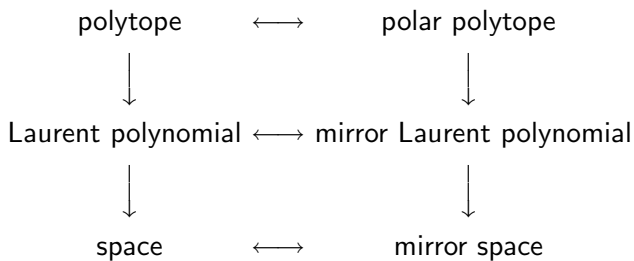
Definition

A lattice polytope Δ is **reflexive** if Δ° is also a lattice polytope.

- ▶ If Δ is reflexive, $(\Delta^\circ)^\circ = \Delta$.
- ▶ Δ and Δ° are a **mirror pair**.



Mirror Polytopes Yield Mirror Spaces



From Polytopes to Polynomials

- ▶ Standard basis vectors in $N \leftrightarrow$ variables z_i

$$(1, 0, \dots, 0) \leftrightarrow z_1$$

$$(0, 1, \dots, 0) \leftrightarrow z_2$$

...

$$(0, 0, \dots, 1) \leftrightarrow z_n$$

- ▶ Lattice points in $\Delta^\circ \leftrightarrow$ monomials defined on $(\mathbb{C}^*)^n$

$$(m_1, \dots, m_k) \leftrightarrow$$

$$z_1^{(1,0,\dots,0) \cdot (m_1,\dots,m_k)} z_2^{(0,1,\dots,0) \cdot (m_1,\dots,m_k)} \dots z_k^{(0,0,\dots,1) \cdot (m_1,\dots,m_k)}$$

- ▶ $\Delta^\circ \leftrightarrow$ Laurent polynomials p_α defined on $(\mathbb{C}^*)^n$

From Polynomials to Spaces

The solutions to the Laurent polynomials p_α describe geometric spaces.

- ▶ $-z_1^{-1} + z_1 = 0$
- ▶ $z_1^{-1} + z_1 = 0$
- ▶ $z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1 = 0$

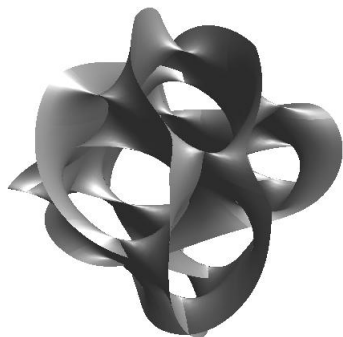


Figure: Slice of a Calabi-Yau threefold

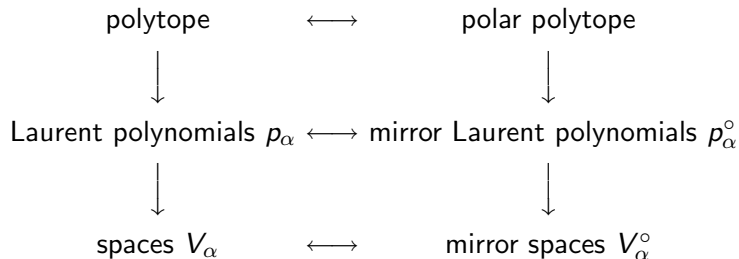
Compactifying

Our Laurent polynomials p_α define spaces which are not compact: $\|z_i\|$ can be infinitely large. We can solve this problem by adding in some “points at infinity” using a standard procedure from algebraic geometry.

The resulting compact spaces V_α are Calabi-Yau varieties of dimension $d = k - 1$.

- ▶ When $k = 2$, for generic choice of α , the V_α are elliptic curves.
- ▶ When $k = 4$, for generic choice of α , the V_α are smooth **3-dimensional Calabi-Yau manifolds**.

Mirror Symmetry



Counting Complex Moduli

The possible deformations of complex structure of V_α form a complex vector space of dimension $h^{d-1,1}(V_\alpha)$.

For $k \geq 4$,

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ) \ell^*(\hat{\Theta}^\circ)$$

- ▶ $\ell()$ = number of lattice points
- ▶ $\ell^*()$ = number of lattice points in the relative interior of a polytope or face
- ▶ The Γ° are codimension 1 faces of Δ°
- ▶ The Θ° are codimension 2 faces of Θ°
- ▶ $\hat{\Theta}^\circ$ is the face of Δ dual to Θ°

Counting Kähler Moduli

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta) \ell^*(\hat{\Theta})$$

- ▶ $\ell()$ = number of lattice points
- ▶ $\ell^*()$ = number of lattice points in the relative interior of a polytope or face
- ▶ The Γ are codimension 1 faces of Δ
- ▶ The Θ are codimension 2 faces of Δ
- ▶ $\hat{\Theta}$ is the face of Δ dual to Θ

Comparing V and V°

For $k \geq 4$,

$$h^{1,1}(V_\alpha) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta) \ell^*(\hat{\Theta})$$

$$h^{d-1,1}(V_\alpha) = \ell(\Delta^\circ) - k - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ) \ell^*(\hat{\Theta}^\circ)$$

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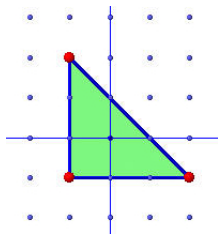
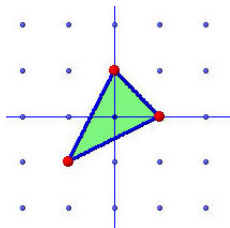
$$h^{d-1,1}(V_\alpha^\circ) = \ell(\Delta) - k - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta) \ell^*(\hat{\Theta})$$

Mirror Symmetry from Mirror Polytopes

We have mirror families of Calabi-Yau varieties V_α and V_α° of dimension $d = k - 1$.

$$h^{1,1}(V_\alpha) = h^{d-1,1}(V_\alpha^\circ)$$
$$h^{d-1,1}(V_\alpha) = h^{1,1}(V_\alpha^\circ)$$

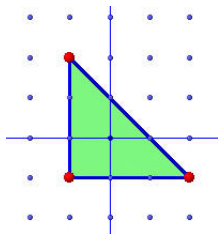
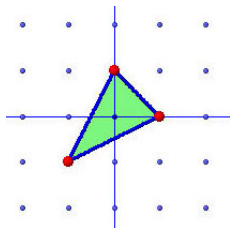
An Example



Four-dimensional analogue:

- ▶ Δ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- ▶ Δ° has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

An Example



Four-dimensional analogue:

- ▶ Δ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- ▶ Δ° has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

$$\begin{aligned}h^{1,1}(V_\alpha) &= \ell(\Delta) - n - 1 - \sum_{\Gamma} \ell^*(\Gamma) + \sum_{\Theta} \ell^*(\Theta) \ell^*(\hat{\Theta}) \\ &= 6 - 4 - 1 - 0 - 0 = 1.\end{aligned}$$

Example (Continued)

- ▶ Δ has vertices $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, $(0, 0, 0, 1)$, and $(-1, -1, -1, -1)$.
- ▶ Δ° has vertices $(-1, -1, -1, -1)$, $(4, -1, -1, -1)$, $(-1, 4, -1, -1)$, $(-1, -1, 4, -1)$, and $(-1, -1, -1, 4)$.

$$h^{1,1}(V_\alpha) = 1$$

$$\begin{aligned} h^{3-1,1}(V_\alpha) &= \ell(\Delta^\circ) - n - 1 - \sum_{\Gamma^\circ} \ell^*(\Gamma^\circ) + \sum_{\Theta^\circ} \ell^*(\Theta^\circ) \ell^*(\hat{\Theta}^\circ) \\ &= 126 - 4 - 1 - 20 - 0 = 101. \end{aligned}$$

The Hodge Diamond

Calabi-Yau Threefolds

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & & & \\ & & & & 0 & & \\ & & & & & & 0 \\ & & & & h^{1,1}(V) & & 0 \\ 1 & & 0 & & & & h^{2,1}(V) & & 1 \\ & & 0 & & h^{1,1}(V) & & h^{2,1}(V) & & 0 \\ & & & & & & & & 0 \\ & & & & 0 & & & & \\ & & & & & & & & \\ & & & & 1 & & & & \end{array}$$

The Hodge Diamond

Calabi-Yau Threefolds

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & & & \\ & & & 0 & & 0 & \\ & & 0 & & h^{1,1}(V) & & 0 \\ 1 & & h^{2,1}(V) & & h^{1,1}(V) & & h^{2,1}(V) \\ & 0 & & & h^{2,1}(V) & & 1 \\ & & 0 & & h^{1,1}(V) & & 0 \\ & & & & 0 & & \\ & & & & 1 & & \end{array}$$

V_α

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & & & \\ & & & 0 & & 0 & \\ & & 0 & & 1 & & 0 \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & & 1 & & 0 \\ & & 0 & & 0 & & \\ & & & & 1 & & \end{array}$$

V_α°

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & & & \\ & & & 0 & & 0 & \\ & & 0 & & 101 & & 0 \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & & 101 & & 0 \\ & & 0 & & 0 & & \\ & & & & 1 & & \end{array}$$