

Reflexive Polytopes and Mirror Symmetry

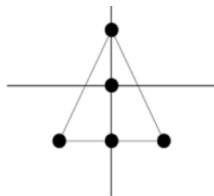
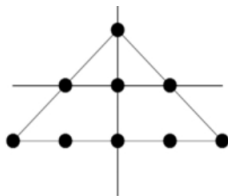
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Lattice Polygons

The points in the plane with integer coordinates form a *lattice* M . A *lattice polygon* is a polygon in the plane which has vertices in the lattice.



Reflexive Polygons

We say a lattice polygon is *reflexive* if it has only one lattice point, the origin, in its interior.

Question

How many reflexive polygons are there?

Infinite Families

There are infinite families of reflexive polytopes.
For instance, the map

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

yields an infinite family of triangles.

Symmetries of M

Definition

$\mathbf{GL}(2, \mathbb{Z})$ is the set of two-by-two matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which have integer entries and determinant $ad - bc$ equal to either 1 or -1 .

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a matrix in $\mathbf{GL}(2, \mathbb{Z})$, the map m_A given by

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

is a one-to-one and onto map of the plane which sends M to M . In particular, m_A sends reflexive polygons to reflexive polygons.

Equivalence Classes

We say two reflexive polygons Γ and Δ are $\mathbf{GL}(2, \mathbb{Z})$ -*equivalent* (or sometimes just *equivalent*) if there exists a matrix $A \in \mathbf{GL}(2, \mathbb{Z})$ such that $m_A(\Gamma) = \Delta$.

Question

How many $\mathbf{GL}(2, \mathbb{Z})$ equivalence classes of reflexive polygons are there?

A Dual Lattice

Let N be another copy of the points in the plane with integer coordinates.

The dot product lets us pair points in M with points in N :

$$(m_1, m_2) \cdot (n_1, n_2) = m_1 n_1 + m_2 n_2$$

Polar Polygons

Let Δ be a lattice polygon in M which contains $(0, 0)$. The *polar polygon* Δ^0 is the polygon in N given by:

$$\{(n_1, n_2) : (m_1, m_2) \cdot (n_1, n_2) \geq -1 \text{ for all } (m_1, m_2) \in \Delta\}$$

Mirror Pairs

If Δ is a reflexive polygon, then:

- ▶ Δ^0 is also a reflexive polygon
- ▶ $(\Delta^0)^0 = \Delta$.

Δ and Δ^0 are a *mirror pair*.

Other Dimensions

Definition

Let $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_q\}$ be a set of points in \mathbb{R}^k . The *polytope* with vertices $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_q\}$ is the set of points of the form

$$\vec{x} = \sum_{i=1}^q t_i \vec{v}_i$$

where the t_i are nonnegative real numbers and $t_1 + t_2 + \dots + t_q = 1$.

Polar Polytopes

Let M be the lattice of points with integer coordinates in \mathbb{R}^k . A *lattice polytope* has vertices in M .

As before, we have a *dual lattice* N and a dot product

$$(m_1, \dots, m_k) \cdot (n_1, \dots, n_k) = m_1 n_1 + \dots + m_k n_k$$

Definition

Let Δ be a lattice polygon in M which contains $(0, 0)$. The *polar polytope* Δ^0 is the polytope in N given by:

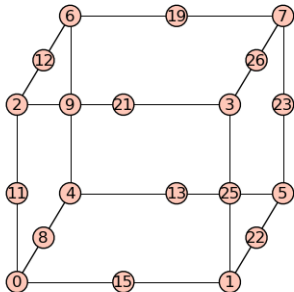
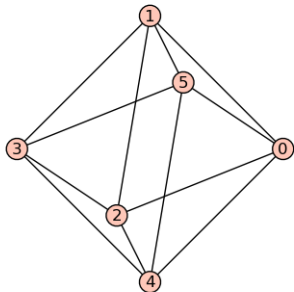
$$\{(n_1, \dots, n_k) : (m_1, \dots, m_n) \cdot (n_1, \dots, n_k) \geq -1 \text{ for all } (m_1, m_2) \in \Delta\}$$

Reflexive Polytopes

Definition

A lattice polytope Δ is *reflexive* if Δ^0 is also a lattice polytope.

- ▶ If Δ is reflexive, $(\Delta^0)^0 = \Delta$.
- ▶ Δ and Δ^0 are a *mirror pair*.



How Many?

Up to equivalence, there are . . .

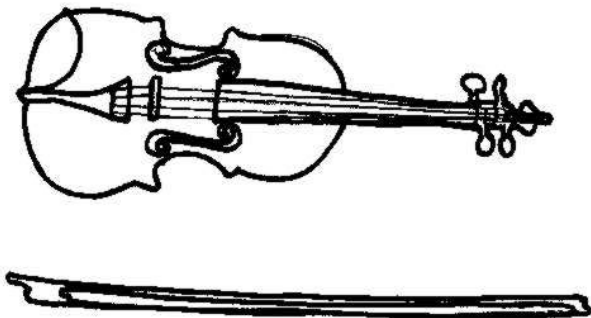
Dimension	Reflexive Polytopes
1	1
2	16
3	4319
4	473,800,776
5	??

Why Do Physicists Care?

- ▶ We understand gravity on a large spatial scale (planets, stars, galaxies).
- ▶ We understand quantum physics on a small spatial scale (electrons, photons, quarks).

Are Strings the Answer?

- ▶ “Fundamental” particles are *strings* vibrating at different frequencies.



- ▶ Strings *wrap* other dimensions!

An Unexpected Duality

Different shapes for the extra dimensions yield the same physics.

Question

How can we describe these shapes?

Mirror Polytopes, Mirror Families

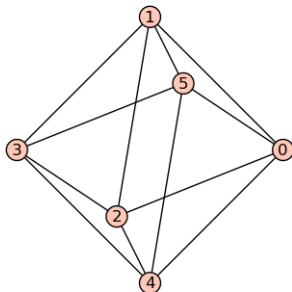
- ▶ Lattice points in $\Delta^0 \leftrightarrow$ monomials in $(\mathbb{C}^*)^n$

$$(n_1, \dots, n_k) \mapsto z_1^{(1,0,\dots,0) \cdot (n_1,\dots,n_k)} z_2^{(0,1,\dots,0) \cdot (n_1,\dots,n_k)} \dots z_k^{(0,0,\dots,1) \cdot (n_1,\dots,n_k)}$$

- ▶ $\Delta^0 \leftrightarrow$ polynomials in $(\mathbb{C}^*)^n$

Research Directions

- ▶ Symmetric polytopes, symmetric polynomials



- ▶ Higher-dimensional reflexive polytopes?