From Polygons to String Theory

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In this article, we will introduce you to a special kind of polygon called a reflexive polygon and to higher-dimensional reflexive polytopes. Two theoretical physicists, Maximilian Kreuzer and Harald Skarke, worked out a detailed description of three- and four-dimensional reflexive polytopes in the late 1990s. Why were two turn-of-the-millennium physicists studying special polyhedra and polytopes? Their motivation came from string theory: using polytopes, physicists were able to construct geometric spaces that could model extra dimensions of our universe. A simple relationship between "mirror" pairs of polytopes corresponds to an extremely subtle connection between these geometric spaces. The quest to understand this connection has created the thriving field of mathematical research known as mirror symmetry.

What are reflexive polytopes? What does it mean to classify them? Why do they come in pairs? How can we build a complicated geometric space from a simple object like a triangle or a cube? And what does any of this have to do with physics? By answering these questions, we will uncover intricate relationships between combinatorics, geometry, and modern physics.

Reflexive Polygons

Classifying Reflexive Polygons

The points in the plane with integer coordinates form a *lattice*, which we'll name N. A *lattice polygon* is a polygon in the plane which has vertices in the lattice; in other words, lattice polygons have vertices with integer coordinates. We assume that all of our polygons are *convex*, that is, every straight-line path between points in the polygon is contained in the polygon. An example lattice polygon is illustrated in Figure 1.

We say a lattice polygon is *reflexive* if it has only one lattice point, the origin, in its interior.

How many reflexive polygons are there? Can we list them all? The first step is to pull out your graph paper and try to draw a reflexive polygon! A little contemplation will produce several reflexive polygons, including triangles, quadrilaterals, and hexagons. Some examples are shown in Figures 2, 3, and 4.

There are also ways to make new reflexive polygons, once you have found your first reflexive polygon. For instance, you can rotate by 90 degrees or reflect across the x-axis. A more complicated type of map is the shear, which stretches a polygon in one direction. We can describe the shear using matrix multiplication: we map the point $\begin{pmatrix} x \\ y \end{pmatrix}$ to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix}.$$

In Figures 5 and 6, we illustrate the effects of this shear on a reflexive triangle. Notice that after the shear, there is still only one point in the interior of our triangle. Repeating the shear map, as seen in Figure 7, makes

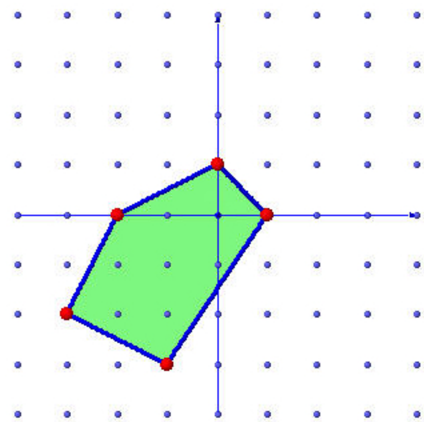

Figure 1: A lattice polygon

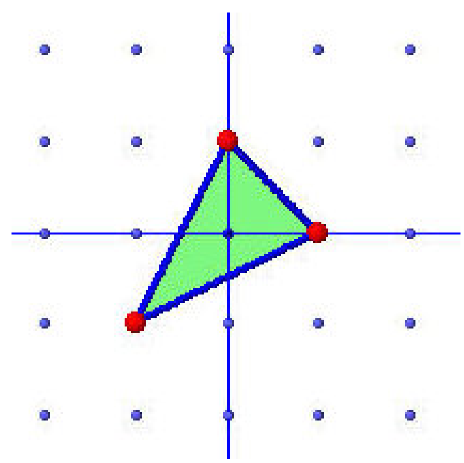

Figure 2: A reflexive triangle

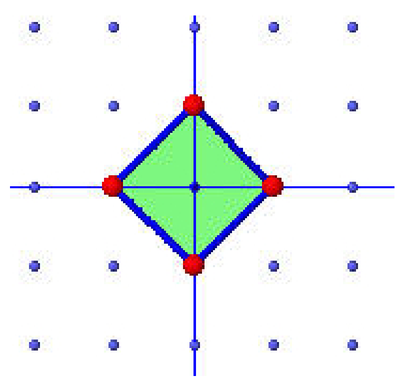

Figure 3: A reflexive quadrilateral

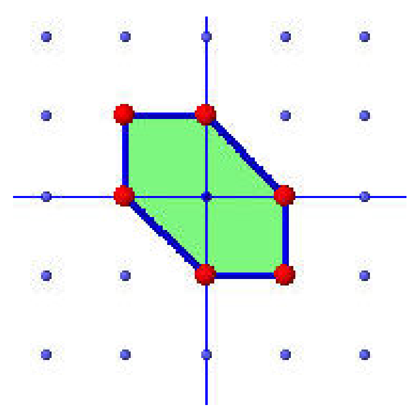

Figure 4: A reflexive hexagon

our triangle longer and skinnier. Iterating the shear map produces an infinite family of reflexive triangles, each one longer and skinner than the one before it.

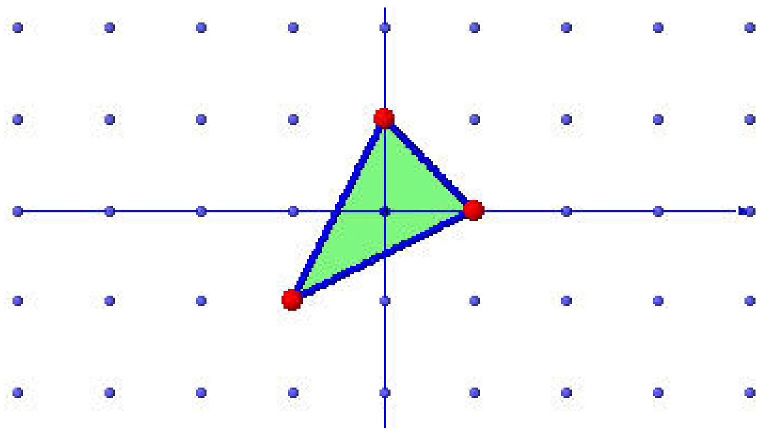

Figure 5: Our starting triangle

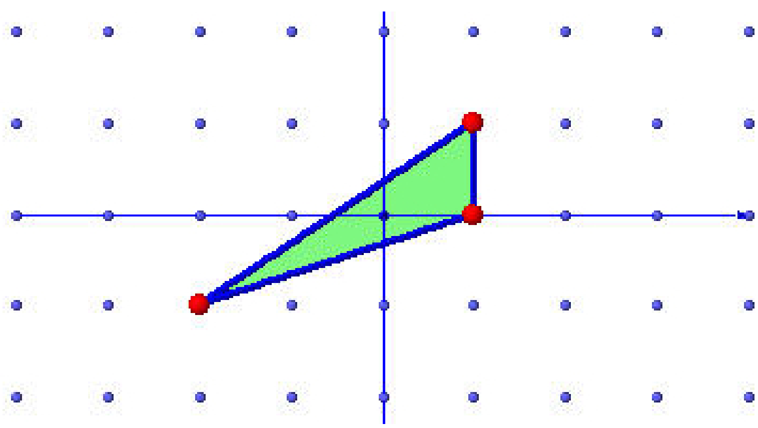

Figure 6: Triangle after shear map

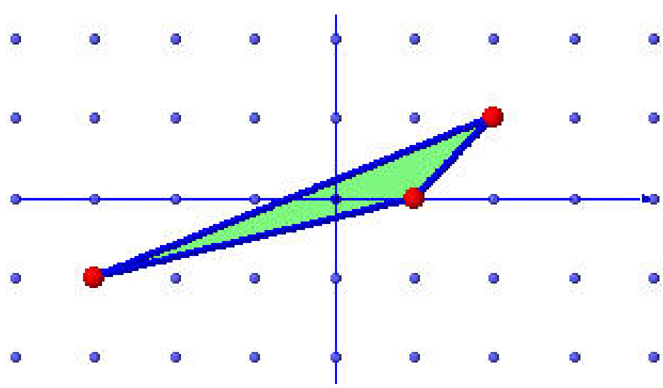

Figure 7: Triangle after two shears

Listing all reflexive polygons is an impossible task! But we would still like to classify reflexive polygons in some way. To do so, we shift our focus: instead of counting individual reflexive polygons, we will count types or classes of reflexive polygons. We want two reflexive polygons to belong to the same equivalence class if you can get from one to the other using reflections, rotations, and shears. Each map that we use should send lattice polygons to other lattice polygons. We can ensure that lattice polygons go to other lattice polygons by

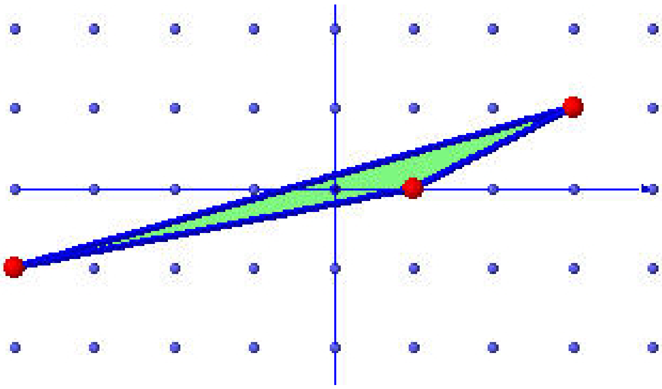

Figure 8: Triangle after three shears

requiring that our reflections, rotations, and shears map the lattice N to itself.

We can describe maps that send N to itself using two-by-two matrices. The matrices need integer entries, so that they will map points with integer coordinates to points with integer coordinates. The matrices need to map the plane to itself, so they have to be invertible. And we want the image of each map to contain all of the points with integer coordinates, so the determinant of the corresponding matrix has to be 1 or -1. (It follows that our matrices will preserve the area of lattice polytopes.)

Matrices with these properties have a special name:

Definition. $\mathbf{GL}(2,\mathbb{Z})$ is the set of two-by-two matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ which have integer entries and determinant $ad - bc$ equal to either 1 or -1.

Because of the determinant condition, the inverse of a matrix in $\mathbf{GL}(2,\mathbb{Z})$ is another matrix in $\mathbf{GL}(2,\mathbb{Z})$. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a matrix in $\mathbf{GL}(2,\mathbb{Z})$. Let m_A be the "multiplication by A" map:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

We see that m_A sends a point in the plane to another point in the plane. Because A belongs to $\mathbf{GL}(2,\mathbb{Z})$, the map m_A is a continuous, one-to-one, and onto map of the plane which sends the lattice N to itself. The map sends lattice polygons to lattice polygons. In particular, the origin is the only lattice point in the interior of a reflexive polygon, so it is the only lattice point which can be mapped to the interior of the image of a reflexive polygon. Thus, this map sends reflexive polygons to reflexive polygons.

Definition. We say two reflexive polygons Δ and Δ' are $\mathbf{GL}(2,\mathbb{Z})$-*equivalent* (or sometimes just *equivalent*) if there exists a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathbf{GL}(2,\mathbb{Z})$ such that $m_A(\Delta) = \Delta'$.

Now that we have a concept of equivalent reflexive polygons, we can try to describe the possible reflexive polygons again. How many $\mathbf{GL}(2,\mathbb{Z})$ equivalence classes of reflexive polygons are there?

It turns out that there are only sixteen equivalence classes of reflexive polygons! (Several proofs of this fact are sketched in [5].) A representative from each reflexive polygon equivalence class is shown in Figure 10.

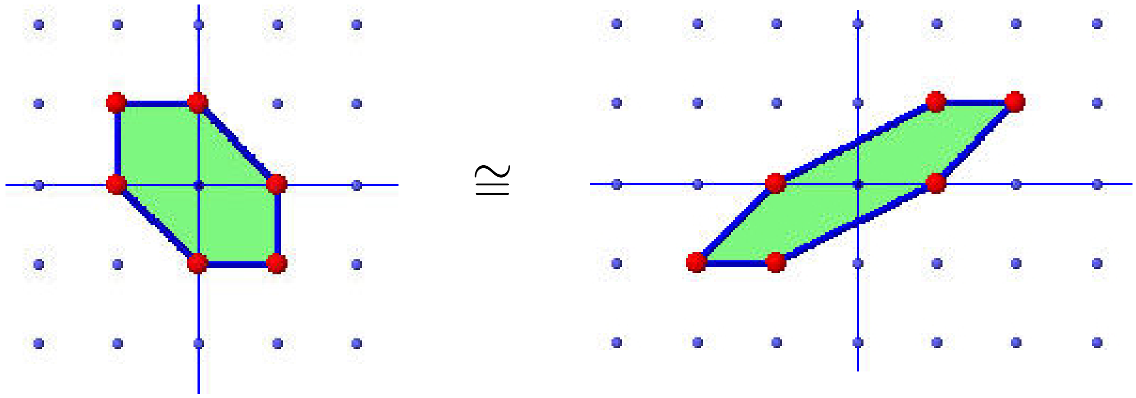

Figure 9: Two equivalent hexagons

Polar Polygons

The vertical arrows in Figure 10 indicate a relationship between pairs of reflexive polygons. In this section, we explain the correspondence. We start by asking a simple question: how can we describe a reflexive polygon mathematically?

One way to describe a polygon is to list the vertices. For instance, for the triangle in Figure 11, the vertices are $(0,1)$, $(1,0)$, and $(-1,-1)$.

Each edge of a polygon is part of a line, so we can also describe a polygon by listing the equations of these lines. For the triangle in Figure 11, the equations are:

$$\begin{aligned} -x-y &= -1 \\ 2x-y &= -1 \\ -x+2y &= -1. \end{aligned}$$

Of course, there are many equivalent ways to write the equation for a line. We chose $ax+by=-1$ as our standard form. (Any line that does not pass through the origin can be written in this way.) Our standard form has the advantage that the whole triangle is described as the set of points (x,y) such that

$$\begin{aligned} -x-y &\geq -1 \\ 2x-y &\geq -1 \\ -x+2y &\geq -1. \end{aligned}$$

Notice that in the case of our reflexive triangle, the coefficients a and b in our standard form for the equation of a line were always integers. This will hold for any reflexive polygon.

We want to use our edge equations to define a new reflexive polygon. We'd like our new polygon to live in its own copy of the plane. Let's call the points in this new plane which have integer coordinates M, and name the new plane $M_{\mathbb{R}}$. Using the dot product, we can combine a point in our old plane, $N_{\mathbb{R}}$, with a point in our new plane, $M_{\mathbb{R}}$, to produce a real number:

$$(n_1,n_2)\cdot(m_1,m_2) = n_1m_1 + n_2m_2.$$

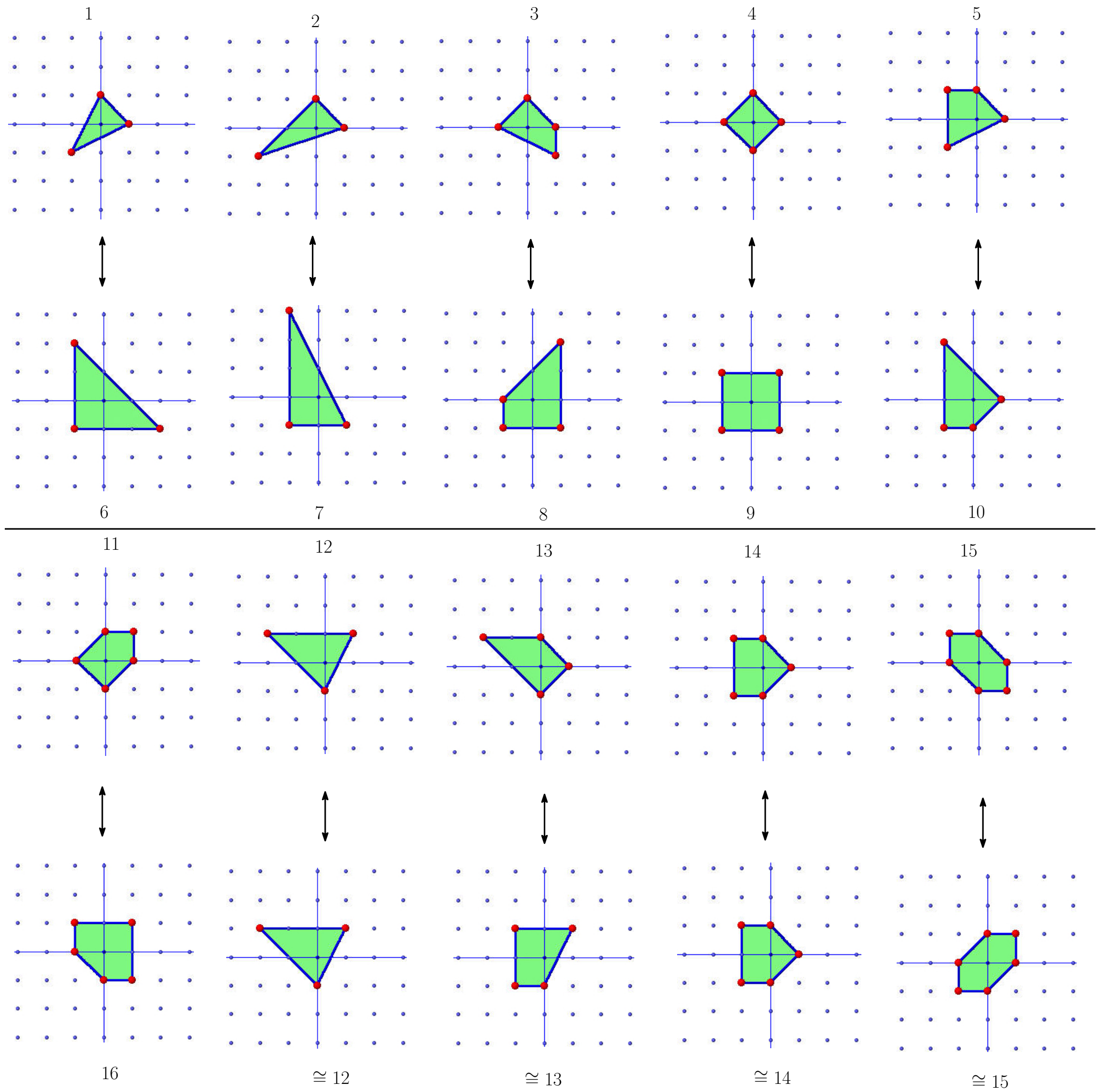

Figure 10: Classification of reflexive polygons

If the point (n_1, n_2) lies in N and the point (m_1, m_2) lies in M, their dot product $(n_1, n_2) \cdot (m_1, m_2)$ will be an integer.

Let's rewrite the edge equations of our reflexive triangle using dot product notation:

$$\begin{aligned} (x, y) \cdot (-1, -1) &= -1 \\ (x, y) \cdot (2, -1) &= -1 \\ (x, y) \cdot (-1, 2) &= -1 \end{aligned}$$

The points $(-1, -1)$, $(2, -1)$, and $(-1, 2)$ are the vertices of a new triangle in $M_{\mathbb{R}}$! We say that the new triangle, shown in Figure 12, is the *polar polygon* of our original triangle.

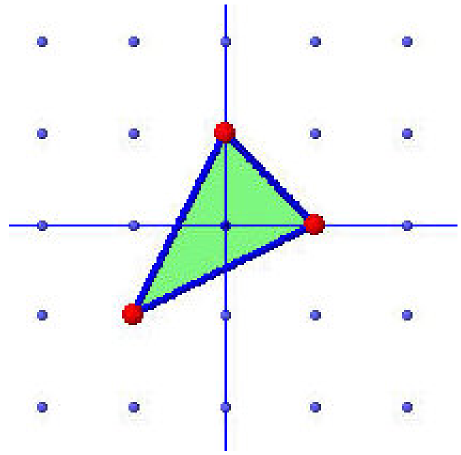

Figure 11: Our reflexive triangle

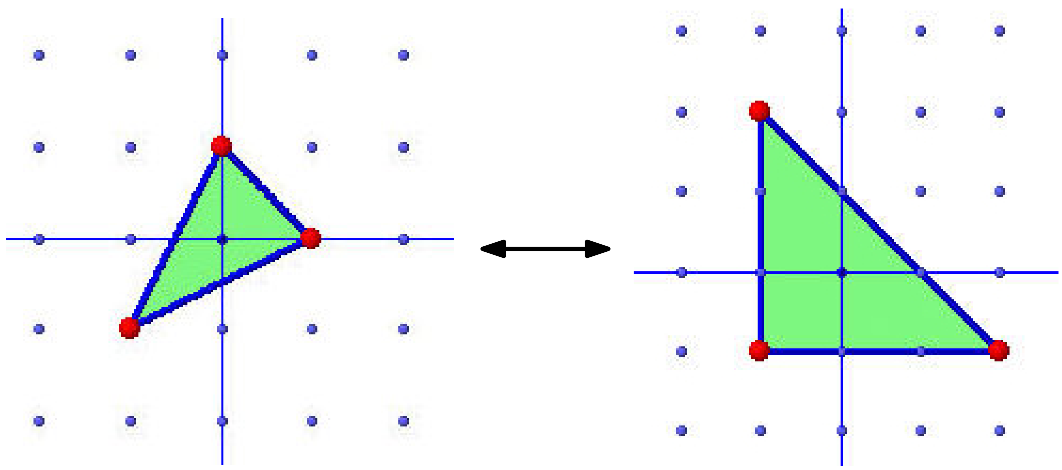

Figure 12: Our reflexive triangle and its polar polygon

Now, let Δ be any lattice polygon in our original plane $N_{\mathbb{R}}$ which contains $(0,0)$. Formally, we say that the *polar polygon* Δ^0 is the polygon in $M_{\mathbb{R}}$ consisting of the points (m_1, m_2) such that

$$(n_1, n_2) \cdot (m_1, m_2) \geq -1$$

for all points (n_1, n_2) in Δ. We can find the vertices of Δ^0 using our standard-form equations for the edges of Δ, just as we did above.

If Δ is a reflexive polygon, then its polar polygon Δ^0 is also a reflexive polygon. We can repeat the polar polygon construction to find the polar polygon of Δ^0. It turns out that the result is our original polygon: $(\Delta^0)^0 = \Delta$. (For practice, try checking this assertion by finding the polar polygon of the second triangle in Figure 12.) We say that a polygon Δ and its polar polygon Δ^0 are a *mirror pair*. The term *reflexive* also refers to this property: in a metaphorical sense, Δ and its polar polygon Δ^0 are reflections of each other. The vertical arrows in Figure 10 connect each reflexive polygon to its polar dual. Notice that some reflexive polygons, such as the hexagon and pentagon numbered 14 and 15 in Figure 10, are self-dual: their polar duals are equivalent to the original polygon.

A reflexive polygon and its polar dual are intricately related. It's pretty easy to see that a polygon and its polar dual will have the same number of sides and vertices. Other connections are more subtle. For instance, the number of lattice points on the boundary of a reflexive polygon and the number of lattice points on the boundary of its polar dual always add up to twelve! For the polygons in Figure 12, the computation is $3 + 9 = 12$. (See [5] for proofs of this fact using combinatorics, algebraic geometry, and number theory.)

Higher Dimensions

Let's extend the idea of reflexive polygons to dimensions other than 2. In order to do so, we need to describe the k-dimensional generalizations of polygons, which we will call *polytopes*. There are several ways to do this. We take the point of view that polygons are described by writing down a list of vertices, adding line segments that connect these vertices, and then filling in the interior of the polygon. Similarly, in k dimensions our intuition suggests that we should describe a polytope by writing down a list of vertices, connecting them, and then filling in the inside. The formal definition is as follows.

Definition. Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ be a set of points in $\mathbb{R}^k$. The *polytope* with vertices $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_q\}$ is the set of points of the form

$$\vec{x} = \sum_{i=1}^{q} t_i \vec{v}_i$$

where the t_i are nonnegative real numbers satisfying $t_1 + t_2 + \cdots + t_q = 1$.

We illustrate a three-dimensional polytope in Figure 13.

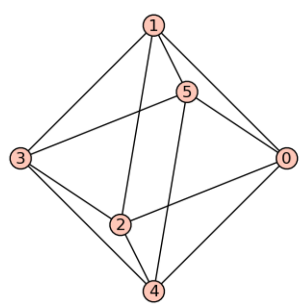

Figure 13: A three-dimensional polytope

Let N be the lattice of points with integer coordinates in $\mathbb{R}^k$; we refer to this copy of $\mathbb{R}^k$ as $N_{\mathbb{R}}$. A *lattice polytope* is a polytope whose vertices lie in N. In order to understand the definition of a lattice polytope better, let's try to understand the one-dimensional polytope that has vertices 1 and -1. Formally, this polytope consists of all points on the real number line that can be written as $1 \cdot t_1 + -1 \cdot t_2$, where $t_1 \geq 0$, $t_2 \geq 0$, and $t_1 + t_2 = 1$. We can visualize this by imagining an ant walking on the numberline. Our ant starts at 0; for a fraction of an hour the ant walks to the right, toward the point 1, then for the rest of the hour the ant walks left, toward the point -1. The ant can reach any point in the closed interval $[-1, 1]$, so all of these points belong to our lattice polytope. This one-dimensional lattice polytope is shown in Figure 14.

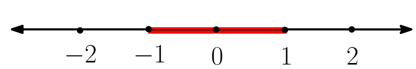

Figure 14: A one-dimensional polytope

Just as we did in two dimensions, we can define a *dual lattice* M in k dimensions by taking a new copy of $\mathbb{R}^k$, which we'll refer to as $M_{\mathbb{R}}$, and letting M be the points with integer coordinates in $M_{\mathbb{R}}$. The dot product pairs points in $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ to produce real numbers:

$$(n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) = n_1 m_1 + \cdots + n_k m_k$$

If we take the dot product of a point in our original lattice N and a point in our dual lattice M, we obtain an integer.

We can use our k-dimensional dot product to define polar polytopes. If Δ is a lattice polytope in N which contains the origin, we say its *polar polytope* Δ^0 is the polytope in M given by:

$$\{(m_1, \ldots, m_k) : (n_1, \ldots, n_k) \cdot (m_1, \ldots, m_k) \geq -1 \text{ for all } (n_1, n_2) \in \Delta\}$$

We defined reflexive polygons as lattice polygons with the property that they have only one lattice point, the origin, in their interior. Later, we found that reflexive polygons also have the property that their polar polygon is another a lattice polygon. In dimensions three and higher, these two properties are not equivalent: there are some lattice polytopes which contain only one lattice point, the origin, in their interior, but whose polar polytope is not a lattice polytope. We take the second property as our definition of a reflexive polytope in an arbitrary dimension: that is, we say that a lattice polytope Δ is *reflexive* if Δ^0 is also a lattice polytope. Just as in two dimensions, we find that the polar of the polar of a reflexive polytope is the original polytope ($(\Delta^0)^0 = \Delta$), and we say that a reflexive polytope Δ and its polar polytope Δ^0 are a *mirror pair.* We illustrate a mirror pair of three-dimensional polytopes in Figure 15.

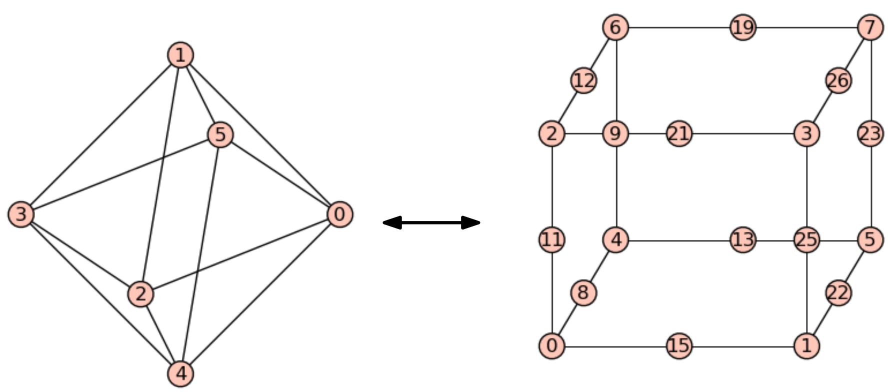

Figure 15: The octahedron and cube: a mirror pair

How many equivalence classes of reflexive polytopes are there in dimension n? It's easy to see that in one dimension there is only one reflexive polytope, namely, the closed interval $[-1, 1]$. (It follows that the one-dimensional reflexive polytope is its own polar polytope.) We have seen the sixteen equivalence classes of two-dimensional reflexive polygons. Two physicists, Maximilian Kreuzer and Harald Skarke, counted equivalence classes of reflexive polytopes in dimensions three and four. Their results are summarized in Table 1; a description of a representative polytope from each class may be found at [4].

The physicists' method for classifying polytopes was very computationally intensive, so it is not effective in higher dimensions. In dimensions five and higher, the number of equivalence classes of reflexive polytopes is an open problem!

Dimension	Classes of Reflexive Polytopes
1	1
2	16
3	4,319
4	473,800,776
≥ 5	??

Table 1: Counting Reflexive Polytopes

The Connection to String Theory

String Theory and Mirror Families

Why were physicists classifying reflexive polytopes? The answer lies in a surprising prediction made by string theory.

String theory is one candidate for what physicists call a *Grand Unified Theory*, or GUT for short. A Grand Unified Theory would unite the theory of general relativity with the theory of quantum physics. General relativity is an effective description for the way our universe works on a very large scale, at the level of stars, galaxies, and black holes. The theory of quantum physics, on the other hand, describes the way our universe works on a very small scale, at the level of electrons, quarks, and neutrinos. Attempts to combine the theories have failed: standard methods for "quantizing" physical theories don't work when applied to general relativity, because they predict that empty space should hold infinite energy.

String theory solves the infinite energy problem by re-defining what a fundamental particle should look like. We often imagine electrons as point particles, that is, zero-dimensional objects. According to string theory, we should treat the smallest components of our universe as one-dimensional objects called *strings*. Strings can be open, with two endpoints, or they can be closed loops. They can also vibrate with different amounts of energy. The different vibration frequencies produce all the particles that particle physicists observe: quarks, electrons, photons, and so forth.

We think of point particles as located somewhere in four dimensions of space and time. To be consistent, string theory requires that strings extend beyond these familiar four dimensions, into extra dimensions. The extra dimensions must have particular geometric shapes. Mathematically, these shapes are known as *Calabi-Yau manifolds*.

When physicists began to explore the implications of string theory, they stumbled on a surprising correspondence: two different choices for the shape of the extra dimensions of our universe can yield the same observable physics. In more mathematical terms, the correspondence implies that Calabi-Yau manifolds should arise in paired or *mirror* families.

We can use reflexive polytopes to describe mirror families! To do so, we need a recipe that starts with a reflexive polytope and produces a geometric space. We will proceed in two steps: first we will use our polytope to build a family of polynomials, and then we will use our polynomials to describe a family of geometric spaces. (Technically, we will work with Laurent polynomials, which can involve negative powers.) We will obtain a mirror pair of geometric spaces corresponding to each mirror pair of polytopes:

$$\begin{array}{ccc} \text{polytope} & \longleftrightarrow & \text{polar polytope} \\ \big\downarrow & & \big\downarrow \\ \text{Laurent polynomial} & \longleftrightarrow & \text{mirror Laurent polynomial} \\ \big\downarrow & & \big\downarrow \\ \text{family of spaces} & \longleftrightarrow & \text{mirror family} \end{array}$$

From Polytopes to Polynomials

Let Δ be a reflexive polytope. We want to construct a family of polynomials using Δ. We start by defining the variables for our polynomial. We do so by associating the variable z_i to the ith standard basis vector in the lattice N:

$$\begin{aligned} (1, 0, \ldots, 0) &\leftrightarrow z_1 \\ (0, 1, \ldots, 0) &\leftrightarrow z_2 \\ &\cdots \\ (0, 0, \ldots, 1) &\leftrightarrow z_n \end{aligned}$$

We think of the z_i as complex variables: we will let ourselves substitute any nonzero complex number for z_i.

Next, for each lattice point in the polar polytope Δ^0, we define a monomial, using the following rule:

$$(m_1, \ldots, m_k) \leftrightarrow \\ z_1^{(1,0,\ldots,0)\cdot(m_1,\ldots,m_k)} z_2^{(0,1,\ldots,0)\cdot(m_1,\ldots,m_k)} \cdots z_k^{(0,0,\ldots,1)\cdot(m_1,\ldots,m_k)}$$

Finally, we multiply each monomial by a complex parameter α_j, and add up the monomials. This gives us a family of polynomials parameterized by the α_j.

Let's work out what this step looks like in the case of the one-dimensional reflexive polytope $\Delta = [-1, 1]$. Because we are working with a one-dimensional lattice N, there is only one standard basis vector, namely 1. Corresponding to this basis vector, we have one monomial, z_1. Next we consider the polar polytope Δ^0. The one-dimensional reflexive polytope is its own polar dual, so $\Delta^0 = [-1, 1]$. Thus, Δ^0 has three lattice points, -1, 0, and 1. From each of these lattice points, we build a monomial, as follows:

$$\begin{aligned} -1 &\mapsto z_1^{1\cdot -1} = z_1^{-1} \\ 0 &\mapsto z_1^{1\cdot 0} = 1 \\ 1 &\mapsto z_1^{1\cdot 1} = z_1 \end{aligned}$$

Finally, we multiply each monomial by a complex parameter and add the results. We obtain the family of Laurent polynomials $\alpha_1 z_1^{-1} + \alpha_2 + \alpha_3 z_1$, which depends on the three parameters α_1, α_2, and α_3. Notice that, because z_1 is raised to a negative power in the first term, we cannot allow z_1 to be zero.

Next, let's look at the family of Laurent polynomials corresponding to the reflexive triangle in Figure 11. We are now working with a two-dimensional polytope, so we have two variables, z_1 and z_2. The polar polygon of our reflexive triangle is shown in Figure 12. It contains ten lattice points (including the origin), so we will have ten monomials.

$$\begin{aligned}
(-1,2) &\mapsto z_1^{(1,0)\cdot(-1,2)} z_2^{(0,1)\cdot(-1,2)} = z_1^{-1} z_2^2 \\
(-1,1) &\mapsto z_1^{(1,0)\cdot(-1,1)} z_2^{(0,1)\cdot(-1,1)} = z_1^{-1} z_2 \\
(0,1) &\mapsto z_1^{(1,0)\cdot(0,1)} z_2^{(0,1)\cdot(0,1)} = z_2 \\
&\dots \\
(2,-1) &\mapsto z_1^{(1,0)\cdot(2,-1)} z_2^{(0,1)\cdot(2,-1)} = z_1^2 z_2^{-1}
\end{aligned}$$

When we multiply each monomial by a complex parameter and add the results, we obtain a family of Laurent polynomials of the form

$$\alpha_1 z_1^{-1} z_2^2 + \alpha_2 z_1^{-1} z_2 + \alpha_3 z_2 + \alpha_4 z_1^{-1} + \alpha_5 + \alpha_6 z_1 + \alpha_7 z_1^{-1} z_2^{-1} + \alpha_8 z_2^{-1} + \alpha_9 z_1 z_2^{-1} + \alpha_{10} z_1^2 z_2^{-1}.$$

The mirror family of polynomials is obtained from the big reflexive triangle in Figure 12. We are still working in two dimensions, so we still need two variables; let's call these w_1 and w_2. The big triangle's polar polygon is the triangle in Figure 11, since the polar of the polar dual of a polygon is the original polygon. Thus, the mirror family of polynomials will only have four terms, corresponding to the four lattice points of the triangle in Figure 11. It is given by

$$\beta_1 w_1^{-1} w_2^{-1} + \beta_2 w_2 + \beta_3 + \beta_4 w_1.$$

From Polynomials to Spaces

If we set a Laurent polynomial equal to zero, the resulting solutions describe a geometric space. Let's look at some examples using our family $\alpha_1 z_1^{-1} + \alpha_2 + \alpha_3 z_1$ obtained from the one-dimensional reflexive polytope. If we set $\alpha_1 = -1$, $\alpha_2 = 0$, and $\alpha_3 = 1$, we obtain the polynomial $-z_1^{-1} + z_1 = 0$. Solving, we find that $z_1^2 = 1$, so the solutions are the pair of points 1 and -1. If we set $\alpha_1 = 1$, $\alpha_2 = 0$, and $\alpha_3 = 1$, we obtain the polynomial $z_1^{-1} + z_1 = 0$. In this case, we find that $z_1^2 = -1$, so the solutions are the pair of points i and $-i$. (Now we see why it is important to work over the complex numbers!)

As we vary the parameters α_1, α_2, and α_3, we will obtain all pairs of nonzero points in the complex plane. Since the one-dimensional reflexive polytope is its own polar dual, the mirror family will also correspond to pairs of nonzero points in the complex plane. These are zero-dimensional geometric spaces inside a one-complex-dimensional ambient space. To describe more interesting geometric spaces, we'll have to increase dimensions.

What are the spaces corresponding to the mirror pair of triangles in Figure 12? Let's set the Laurent polynomials corresponding to the smaller triangle equal to zero.

$$\alpha_1 z_1^{-1} z_2^2 + \alpha_2 z_1^{-1} z_2 + \alpha_3 z_2 + \alpha_4 z_1^{-1} + \alpha_5 + \alpha_6 z_1 + \alpha_7 z_1^{-1} z_2^{-1} + \alpha_8 z_2^{-1} + \alpha_9 z_1 z_2^{-1} + \alpha_{10} z_1^2 z_2^{-1} = 0.$$

We can multiply through by $z_1 z_2$ without changing the non-zero solutions. We obtain

$$\alpha_1 z_2^3 + \alpha_2 z_2^2 + \alpha_3 z_1 z_2^2 + \alpha_4 z_2 + \alpha_5 z_1 z_2 + \alpha_6 z_1^2 z_2 + \alpha_7 + \alpha_8 z_1 + \alpha_9 z_1^2 + \alpha_{10} z_1^3 = 0.$$

Let's re-order, so that terms of higher degree come first:

$$\alpha_{10}z_1^3 + \alpha_1 z_2^3 + \alpha_6 z_1^2 z_2 + \alpha_3 z_1 z_2^2 + \alpha_9 z_1^2 + \alpha_2 z_2^2 + \alpha_5 z_1 z_2 + \alpha_8 z_1 + \alpha_4 z_2 + \alpha_7 = 0.$$

As we vary our parameters α_i, we see that we will obtain all possible degree-three or *cubic* polynomials in two complex variables. We cannot graph the solutions to these polynomials, because they naturally live in two complex (or four real) dimensions. However, we can graph the solutions that happen to be pairs of real numbers. These will trace out a curve in the plane. The real solutions for two possible choices of the parameters α_i are shown in Figures 16 and 17.

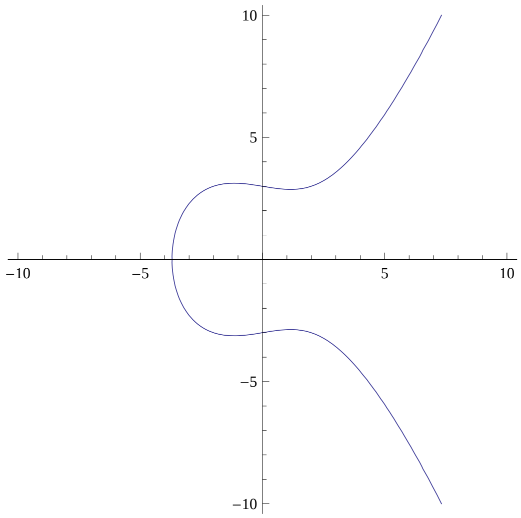

Figure 16: A real cubic curve

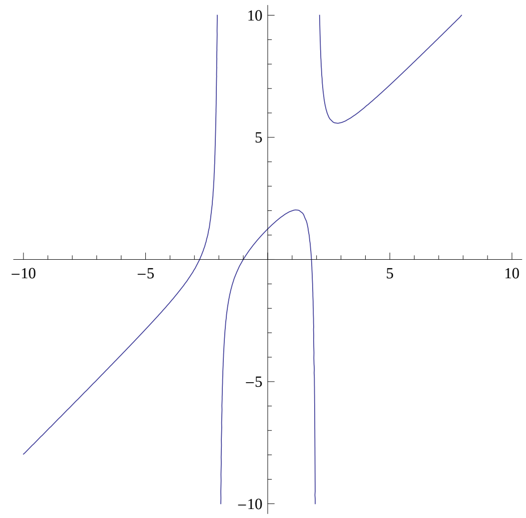

Figure 17: Another cubic curve

The mirror family of spaces is given by solutions to

$$\beta_1 w_1^{-1} w_2^{-1} + \beta_2 w_2 + \beta_3 + \beta_4 w_1 = 0.$$

We can multiply through by $w_1 w_2$ without changing the nonzero solutions:

$$\beta_1 + \beta_2 w_1 w_2^2 + \beta_3 w_1 w_2 + \beta_4 w_1^2 w_2 = 0.$$

Our mirror family of spaces also consists of solutions to cubic polynomials, but instead of taking all possible cubic polynomials, we have a special subfamily.

Physicists are particularly interested in *Calabi-Yau threefolds*: these three complex-dimensional (or six real-dimensional) spaces are candidates for the extra dimensions of the universe. We can generate Calabi-Yau threefolds using reflexive polytopes. For instance, one of the spaces in the family corresponding to the polytope with vertices $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$, and $(-1,-1,-1,-1)$ can be described by the polynomial

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1 = 0.$$

Although we cannot graph this six-dimensional space, we can begin to understand its complexity by drawing a two-dimensional slice in $\mathbb{R}^3$. One possible slice is shown in Figure 18. You can generate and rotate slices of this space at the website [3].

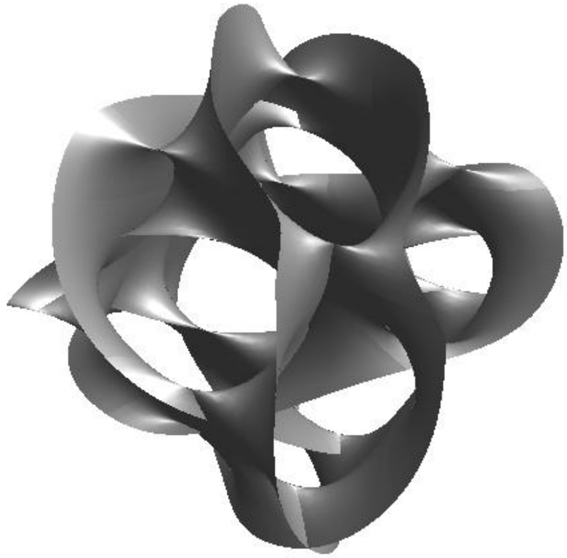

Figure 18: Slice of a Calabi-Yau threefold

Physical, combinatorial, and geometrical dualities

String theory inspired physicists to study the geometric spaces known as Calabi-Yau manifolds. Using the duality between pairs of reflexive polytopes, we have written a recipe for constructing mirror families of these manifolds:

$$\begin{array}{ccc} \text{polytope} & \longleftrightarrow & \text{polar polytope} \\ \Big\downarrow & & \Big\downarrow \\ \text{Laurent polynomials} & \longleftrightarrow & \text{mirror Laurent polynomials} \\ \Big\downarrow & & \Big\downarrow \\ \text{spaces} & \longleftrightarrow & \text{mirror spaces} \end{array}$$

The ingredients in our recipe are combinatorial data, such as the number of points in a lattice polytope; the results of our recipe are paired geometric spaces. But combinatorics not only allows us to cook up these spaces, it gives us a way to study them: we can investigate geometric and topological properties of Calabi-Yau manifolds by measuring the properties of the polytopes we started with. Thus, a straightforward combinatorial duality provides insight into mysterious geometrical dualities (mirror spaces) and even physical dualities (mirror universes)!

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Further Reading

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