Math 136: Complex Variables
Fall 2002
Midterm Exam 2

November 8, 2002

Name: ______________________

Day and Date: ______________ **Start Time:** ___________ **(am/pm)**

Day and Date: ______________ **Finish Time:** ___________ **(am/pm)**

Instructions: This is a closed book, closed notes, 3 hour take home exam. The three hours must be in one continuous block of time. The HMC honour code applies to the taking of this exam. You may use the one page of notes you have prepared. You are not allowed to consult any other notes, or printed source, or people, during the exam, EXCEPT: If you have any questions about the exam, contact Dr. Ward (x76019 or ward@math.hmc.edu).

Justify your answers. Show your work. Partial credit will be given. You may use the results of earlier problems or parts of problems, even if you have not proved them. The exam is due in class at 2:45pm on **Wednesday, 13 November**. I will be happy to accept it earlier; just put it under my office door. If you need an extension, please see me as soon as possible.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

1. (20 points) Short answer question. No proofs required, except where requested.

(a) (3 points) The Cauchy Integral Formula says that under certain circumstances,

$$f(z_0) = \underline{\hspace{6cm}}$$

(b) (4 points) True or false: If $f(z)$ is analytic at each point of a closed contour Γ, then $\int_\Gamma f(z)\,dz = 0$. Prove or give a counterexample.

(c) (3 points) Give an example of a function whose Taylor series diverges at at least one point on the circle of convergence.

(c) (4 points) The residue of $ze^{3/z}$ at $z = 0$ is: ____________

(d) (5 points) What are the first three terms of the Laurent series valid in the annulus $0 < |z| < \infty$ for the function $g(z) = z^2 \cos(1/z)$?

2. (a) (5 points) Find (i) the Taylor series about 0 and (ii) its radius of convergence, for the function

$$f(z) = \frac{z - 1/2}{1 - z/2}.$$

(b) (5 points) (Unrelated to part (a).) Sketch the triangle Γ with vertices at 2, -2, and $2i$, and (without too much computation!) evaluate this integral around Γ, taken counterclockwise:

$$I = \int_\Gamma \frac{e^{z^2}}{(z-i)^3}\, dz.$$

3. (a) (5 points) Prove the Cauchy Estimates: If $f(z)$ is analytic on and inside the circle $C_R : |z - z_0| = R$, and $|f(z)| \leq M$ for all $z \in C_r$, then

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n} \qquad \text{for } n = 1, 2, 3, \ldots$$

(b) (5 points) (Unrelated to part (a).) Prove that if the functions f_n are continuous on a set $S \subset \mathbb{C}$, and if the f_n converge to f uniformly on S, then f is continuous on S.

4. (10 points) Find and classify the singularities of these functions. (Removable, pole, essential.) If there is a pole, find its order. Justify your answers.

(a) $\dfrac{1}{z+(1/z)}$ (b) $\dfrac{1-\cos z}{z^2}$ (c) $\dfrac{1}{\sin\frac{1}{z}}$ (sing. at 0) (d) $\dfrac{1}{\sin\frac{1}{z}}$ (sing. at ∞)

5. (a) (5 points) Suppose $f(z)$ is analytic with a zero of order m at z_0. What kind of singularity does $h(z) = f'(z)/f(z)$ have at z_0? Compute the residue of $h(z)$ at z_0.

(b) (5 points) (Unrelated to part (a).) If f is analytic in the annulus $1 < |z| < 2$, and $|f(z)| < 3$ on $|z| = 1$ and $|f(z)| < 12$ on $|z| = 2$, prove that $|f(z)| < 3|z|^2$ for $1 < |z| < 2$.

6. (10 points) Let C be the unit circle, traversed counterclockwise. Evaluate the integral

$$I = \int_C \frac{e^z}{z(3z-1)^2}\, dz.$$

7. (10 points) Use the Residue Theorem to evaluate the following integral. Justify your work.

$$J = \int_{\infty}^{\infty} \frac{dx}{x^2 - 2x + 5}.$$

8. (10 points) Suppose that $f(z)$ and $g(z)$ are entire, and that $|f(z)| \leq |g(z)|$ for all z. Prove that there is a constant c such that $f(z) = cg(z)$ for all z.

9. (10 points) Let

$$\cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

be the Laurent series for $\cot(\pi z)$ on the annulus $1 < |z| < 2$. Compute the coefficients a_n for all negative n.

Hint: $\cot(\pi z)$ has simple poles at all integers z, with residues π^{-1}, and no other singularities.