Math 136: Complex Variables
Fall 2002
Final Exam

December 14, 2002

Name: ______________________

Day and Date: ______________ **Start Time:** ___________ **(am/pm)**

Finish Time: ___________ **(am/pm)**

Instructions: This is a closed book, closed notes, 6 hour take home exam. The six hours must be in one continuous block of time, except that you may take a half-hour break (during which you may not read or write anything relevant to the exam). The HMC honor code applies to the taking of this exam. You may use the one page of notes you have prepared. You are not allowed to consult any other notes, or printed source, or people, during the exam, except: If you have any questions about the exam, contact Dr. Ward (x76019 or ward@hmc.edu).

Justify your answers, and show your work. Partial credit will be given. You may use the results of earlier parts of problems, even if you have not proved them. The exam is due at my office by 5pm on **Wednesday, 18 December**. I will be happy to accept it earlier; just put it under my office door. If you need an extension, see me as soon as possible.

Question	Points	Score
1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
Total	100	

1. (20 points, 5 points per part)

(a) State the Residue Theorem.

(b) Compute the residue at $z = 1$ of the function $f(z) = \frac{z^2}{(z+1)(z-1)^2}$.

(c) Find the poles and residues of the function $g(z) = 1/(z^2 - 3z + 4)$.

(d) Let a and b be positive numbers. Suppose the line segment from $-b$ to ai makes an angle α with the positive x-direction. Draw the polygon D which is the part of the upper half plane with boundary given by the four line segments $(-\infty, -b]$, $-b$ to ai, ai to b, and $[b, \infty)$. Write down the Schwarz-Christoffel mapping from the upper half plane to D, using $x_1 = -1$, $x_2 = 0$, and $x_3 = 1$.

2. (10 points) Use residue theory to evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{1}{x^6 + 1} \, dx.$$

Justify your work.

3. (10 points) Show that

$$J = \int_0^{2\pi} \frac{1}{3 + 2\cos\theta}\, d\theta = \frac{2\pi}{\sqrt{5}}.$$

Justify your answer.

4. (10 points) Evaluate the integral

$$K = \int_{-\infty}^{\infty} \frac{3x \sin x}{x^2 + 4} \, dx.$$

Justify your answer.

5. (10 points) Show that

$$I = \int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Justify your work.

6. (10 points) Use a Pacman contour to show that

$$J = \int_0^\infty \frac{x^{-1/3}}{x+1}\, dx = \frac{\pi}{\sin(2\pi/3)}.$$

Justify your work.

7. (10 points) Find a Möbius transformation $S(z)$ that maps the open unit disc $\mathbb{D} = \{z \mid |z| < 1\}$ to the open half-plane $R = \{w \mid \mathrm{Re}(w) < -1\}$, and that takes $z = i$ to $w = -1$. Include drawings of the sequence of intermediate domains you use, and give a formula for the overall mapping.

8. (10 points) (a) Find the steady state temperature distribution in the wedge D of angle $\pi/4$, in the first quadrant, bounded by the positive x-axis and by the ray $\arg z = \pi/4$, if the temperature is held constant at -1 along the positive x-axis and at 1 along the ray $\arg z = \pi/4$. (Draw!)

(b) Give equations for the isotherms (level curves of temperature) in D.

9. (10 points) (a) Use residue theory to find the Fourier transform of

$$f(t) = \frac{t}{t^4 + 1}.$$

Hint: Consider the cases $\omega \geq 0$ and $\omega < 0$ separately.

(b) Write down the Fourier inversion formula for $f(t) = t/(t^4 + 1)$. (No need to verify.)