MATH 40 LECTURE 1: VECTORS AND THE DOT PRODUCT

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Our goal in this course is to begin a study of the beautiful world of the linear – linear objects, linear operators, their algebra and even their geometry. Said differently, we hope to study some of the *algebraic* properties of $\mathbb{R}^n$.

The real numbers have a familiar algebraic structure. Given two real numbers a and b, we understand $a + b$, $a \times b$, $a - b$ and a/b (so long as $b \neq 0$ for the latter).

Question 1. *Given two points* $(x_1, x_2, \ldots, x_n)$ *and* $(y_1, y_2, \ldots, y_n)$ *in* $\mathbb{R}^n$*, can I add them? Can I multiply them?*

The answer is yes! Sort of. Well, kind of. We really need to work this out in some detail. To have this discussion it's useful to build some additional terminology. Let's begin with *vectors*.

Definition 1. *A* vector $\vec{v}$ *in* $\mathbb{R}^n$ *is simply a point in* $\mathbb{R}^n$.

Remark 2. We denote the components of $\vec{v}$ by $\vec{v} = (v_1, v_2, v_3, \ldots v_n)$. In other words, a vector in $\mathbb{R}^n$ is a string of n real numbers.

Why in the world do we call them vectors instead of points? Well, one answer is geometric. I think of points as zero dimensional geometric objects. The point $(1, 2, 3) \in \mathbb{R}^3$ is just a dot, sitting at the *location* $(1, 2, 3)$. However, I think of the *vector* $(1, 2, 3)$ as an arrow starting at the origin, ending at $(1, 2, 3)$. This vector has *magnitude* and *direction*.

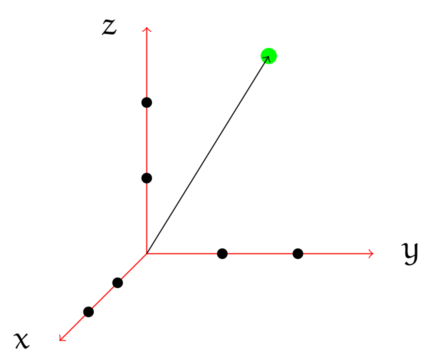

FIGURE 1. The green point $p = (1, 2, 3)$ and the grey vector $\vec{v} = (1, 2, 3)$.

Adding two vectors seems easy enough. We simply add componentwise. If $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$, then

$$\vec{a} + \vec{b} = (a_1 + b_1, \ldots, a_n + b_n).$$

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These are lecture notes for HMC Math 40: Introduction to Linear Algebra and roughly follow our course text *Linear Algebra* by David Poole.

Remark 3. What is the geometric interpretation of vector addition?

We may multiply vectors using the *dot product*.

Definition 4. *The* dot product *of $\vec{a}$ and $\vec{b}$ is defined by*

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Remark 5. Note that $\vec{a} \cdot \vec{b}$ is a simple real number, not a vector.

Definition 6. *To distinguish between elements of $\mathbb{R}$ and vectors in $\mathbb{R}^n$, we call elements of $\mathbb{R}$* scalars*.*

Notice that we can also multiply scalars and vectors. This is creatively termed *scalar multiplication*.

Definition 7. *Let c be a scalar and $\vec{v}$ be a vector. The* scalar multiple *of $\vec{v}$ by c is*

$$c\vec{v} = (cv_1, cv_2, \ldots, cv_n).$$

Remark 8. What is the geometric interpretation of scalar multiplication? Does this shed light on the term "scalar"?

Theorem 9. *Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^n$ and let c be a scalar.*

(1) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
(2) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
(3) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
(4) $\vec{u} \cdot \vec{u} \geqslant 0$ *and* $\vec{u} \cdot \vec{u} = 0$ *if and only if* $\vec{u} = 0$

PROOF. Use the definitions. □

So, this is the dot product. Hooray! Why should we care about it? Is it useful? Does it yield any information?

Definition 10. *The* length *of $\vec{v}$ is defined by*

$$||\vec{v}|| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Theorem 11. *Let $\vec{v}$ be a vector in $\mathbb{R}^n$ and let c be a scalar. Then*

(1) $||\vec{v}|| = 0$ *if and only if* $\vec{v} = \vec{0}$.
(2) $||c\vec{v}|| = |c| \, ||\vec{v}||$.

Theorem 12 (Cauchy-Schwarz Inequality)**.** *Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Then*

$$|\vec{u} \cdot \vec{v}| \leqslant ||\vec{u}|| \, ||\vec{v}||.$$

Theorem 13 (The Triangle Inequality)**.** *Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^n$. Then*

$$||\vec{u} + \vec{v}|| \leqslant ||\vec{u}|| + ||\vec{v}||.$$

PROOF. Study $||\vec{u} + \vec{v}||^2$ and use Cauchy-Schwarz. □

Definition 14. *The* distance *between $\vec{u}$ and $\vec{v}$ is $||\vec{u} - \vec{v}||$.*

Theorem 15. *For nonzero vectors $\vec{u}$ and $\vec{v}$, let θ be the angle between $\vec{u}$ and $\vec{v}$. Then*

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \, ||\vec{v}||}.$$

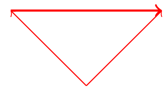

FIGURE 2. The geometry of $\vec{u} - \vec{v}$

PROOF. Apply the law of cosines to Figure 2, to obtain

$$||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2||\vec{u}||\,||\vec{v}||\cos\theta.$$