Suppose \( f \) is continuously differentiable on the interval \([a, b]\).

Let’s derive a formula for the length \( L \) of the curve on the interval, called the arc length over \([a, b]\).

We’ll start by subdividing the interval \([a, b]\) into \( n \) subintervals \([x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n]\) where \( a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \).

Introduce the line segments between \((x_0, f(x_0))\) and \((x_1, f(x_1))\), \((x_1, f(x_1))\) and \((x_2, f(x_2))\), \ldots, \((x_{n-1}, f(x_{n-1}))\) and \((x_n, f(x_n))\).

The resulting polygonal path approximates the curve given by \( y = f(x) \), and its length approximates the arc length of \( f(x) \) over \([a, b]\).

Let’s find the length of the polygonal path by adding up the lengths of the individual line segments. The \( k \)th line segment is the hypotenuse of a triangle with base \( \Delta x_k \) and height \( f(x_k) - f(x_{k-1}) \), and so has length

\[
L_k = \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}.
\]

By the Mean Value Theorem, there exists \( x_k^* \in [x_{k-1}, x_k] \) such that

\[
\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*)
\]

so

\[
f(x_k) - f(x_{k-1}) = f'(x_k^*)(x_k - x_{k-1}) = f'(x_k^*)\Delta x_k.
\]

Thus,

\[
L_k = \sqrt{(\Delta x_k)^2 + [f'(x_k^*)\Delta x_k]^2} = \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k.
\]

Finally, the length of the entire polygonal path is

\[
\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k
\]
which has the form of a **Riemann sum**. Increasing the number of subintervals such that 
\[ \max \Delta x_k \to 0, \sum_{k=1}^{n} L_k \to L. \] 
That is,

\[
L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx
\]

by the definition of the **definite integral** as a limit of Riemann sums. Thus, we have proved
the following:

**Arc Length**

Let \( f(x) \) be continuously differentiable on \([a, b]\). Then the arc length \( L \) of \( f(x) \) over \([a, b]\) is
given by

\[
L = \int_{a}^{b} \sqrt{1 + [f'(x)]^2} \, dx.
\]

Similarly, if \( x = g(y) \) with \( g \) continuously differentiable on \([c, d]\), then the arc length \( L \) of \( g(y) \) over \([c, d]\) is given by

\[
L = \int_{c}^{d} \sqrt{1 + [g'(y)]^2} \, dy.
\]

These integrals often can only be computed using numerical methods.

**Example**

We can compute the arc length of the graph of \( f(x) = x^{3/2} \) over \([0, 1]\) as follows:

\[
L = \int_{0}^{1} \sqrt{1 + [f'(x)]^2} \, dx = \int_{0}^{1} \sqrt{1 + [3x^{1/2}/2]^2} \, dx \\
= \int_{0}^{1} \sqrt{1 + 9x/4} \, dx \\
= \left[ \frac{8}{27} (1 + 9x/4)^{3/2} \right]_{0}^{1} \\
= (1 + 9/4)^{3/2} - (1)^{3/2} \\
= (13/4)^{3/2} - 1 \\
\approx 1.44.
\]

**Exploration**
Key Concepts

Let $f(x)$ be continuously differentiable on $[a, b]$. Then the arc length $L$ of $f(x)$ over $[a, b]$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

Similarly, if $x = g(y)$ with $g$ continuously differentiable on $[c, d]$, then the arc length $L$ of $g(y)$ over $[c, d]$ is given by

$$L = \int_c^d \sqrt{1 + [g'(y)]^2} \, dy$$

[I’m ready to take the quiz.] [I need to review more.] [Take me back to the Tutorial Page]