Harvey Mudd College Math Tutorial:

The Binomial Theorem

We know that

\[(x + y)^0 = 1\]
\[(x + y)^1 = x + y\]
\[(x + y)^2 = x^2 + 2xy + y^2\]

and we can easily expand

\[(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.\]

For higher powers, the expansion gets very tedious by hand! Fortunately, the Binomial Theorem gives us the expansion for any positive integer power of \((x + y)\):

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\]

where

\[
\binom{n}{k} = \frac{(n)(n-1)(n-2)\cdots(n-(k-1))}{k!} = \frac{n!}{k!(n-k)!}
\]

Proof by Induction | Combinatorial Induction | Connection to Pascal's Triangle

**Example**

By the Binomial Theorem,

\[
(x + y)^3 = \sum_{k=0}^{3} \binom{3}{k} x^{3-k} y^k
\]

\[
= \binom{3}{0} x^3 + \binom{3}{1} x^2 y + \binom{3}{2} xy^2 + \binom{3}{3} y^3
\]

\[
= x^3 + 3x^2y + 3xy^2 + y^3
\]

as expected.
Extensions of the Binomial Theorem

A useful special case of the Binomial Theorem is

\[(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\]

for any positive integer \(n\), which is just the Taylor series for \((1 + x)^n\).

This formula can be extended to all real powers \(\alpha\):

\[(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k\]

for any real number \(\alpha\), where

\[\binom{\alpha}{k} = \frac{(\alpha)(\alpha-1)(\alpha-2) \cdots (\alpha-(k-1))}{k!} \frac{\alpha!}{k!(\alpha-k)!} = \frac{\alpha!}{k!(\alpha-k)!}.
\]

Notice that the formula now gives an infinite series. (When \(\alpha = n\) is a positive integer, all but the first \((n+1)\) terms are 0 since after this \(n-n=0\) appears in each numerator.)

This expansion is very useful for approximating \((1 + x)^\alpha\) for \(|x| \ll 1\):

\[(1 + x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} x^3 + \cdots .\]

But for \(|x| \ll 1\), higher powers of \(x\) get small very quickly, so \((1 + x)^\alpha\) can be approximated to any accuracy we need by truncating the series after a finite number of terms.

Example
For \(|x| \ll 1\),

\[(1 + x)^{5/2} \approx 1 + \frac{5}{2} x,\]
\[(1 - 2x)^{100} \approx 1 - 200x,\]
\[(1 + x^2)^{-3} \approx 1 - 3x^2.\]

This type of reasoning is useful in investigating what happens when a physical system is perturbed slightly, introducing a new very small term \(x\).
Key Concepts

Binomial Theorem

For any positive integer $n$,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where

$$\binom{n}{k} = \frac{n(n-1)(n-2) \cdots (n-k+1)}{k!} = \frac{n!}{k!(n-k)!}.$$