Harvey Mudd College Math Tutorial:

Complex Numbers

The complex numbers are an extension of the real numbers containing all roots of quadratic equations. If we define \( i \) to be a solution of the equation \( x^2 = -1 \), then the set \( \mathbb{C} \) of complex numbers is represented in \textbf{standard form} as

\[
\{ a + bi | a, b \in \mathbb{R} \}.
\]

We often use the variable \( z = a + bi \) to represent a complex number. The number \( a \) is called the \textbf{real part} of \( z \) (\( \text{Re} \, z \)) while \( b \) is called the \textbf{imaginary part} of \( z \) (\( \text{Im} \, z \)). Two complex numbers are \textbf{equal} if and only if their real parts are equal and their imaginary parts are equal.

We represent complex numbers graphically by associating \( z = a + bi \) with the point \((a, b)\) on the \textbf{complex plane}.

**Basic Operations**

The basic operations on complex numbers are defined as follows:

\[
(a + bi) + (c + di) = (a + c) + (b + d)i
\]

\[
(a + bi) - (c + di) = (a - c) + (b - d)i
\]

\[
(a + bi)(c + di) = ac + adi + bci + bdi^2
\]

\[
= (ac - bd) + (bc + ad)i
\]

\[
a + bi \quad c + di \quad c - di = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i
\]

In dividing \( a + bi \) by \( c + di \), we rationalized the denominator using the fact that \((c + di)(c - di) = c^2 - cdi + cdi - d^2 i^2 = c^2 + d^2\).

The complex numbers \( c + di \) and \( c - di \) are called \textbf{complex conjugates}. If \( z = c + di \), we use \( \overline{z} \) to denote \( c - di \).

Viewing \( z = a + bi \) as a vector in the complex plane, it has magnitude

\[
|z| = \sqrt{a^2 + b^2},
\]

which we call the \textbf{modulus} or \textbf{absolute value} of \( z \).

Notice that \( zz = |z|^2 \).
Examples

- $(2 + 3i)(2 - 3i) = 4 - 6i + 6i - 9i^2 = 4 + 9 = 13.$
- $|2 + 3i| = |2 - 3i| = \sqrt{4 + 9} = \sqrt{13}.$

Polar Form

For $z = a + bi$, let

\[
\begin{align*}
a &= r \cos \theta \\
b &= r \sin \theta
\end{align*}
\]

from which we can also obtain

\[
\begin{align*}
r &= \sqrt{a^2 + b^2} = |z| \\
\tan \theta &= \frac{b}{a}.
\end{align*}
\]

Then

\[z = r \cos \theta + ir \sin \theta\]

and so, by Euler’s Equation, we obtain the **polar form**

\[z = re^{i\theta}.\]

Here, $r$ is the magnitude of $z$ and $\theta$ is called the **argument** of $z$ (arg $z$). The argument is not unique; we can add multiples of $2\pi$ to $\theta$ without changing $z$. We define Arg $z$, the **principal value** of the argument, to be in $(-\pi, \pi]$. The principal value is unique for each $z$ but creates unavoidable (yet interesting!) complications due to its discontinuity across the negative real axis where it jumps from $\pi$ to $-\pi$. This jump is called a **branch cut**.

Examples

- $e^{i\pi} = \cos \pi + i \sin \pi = -1$
- $3e^{i\pi/2} = 3(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}) = 3i$
- $2e^{i\pi/6} = 2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}) = \sqrt{3} + i$

Multiplication and division of complex numbers is amazingly simple in polar form! If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$, then

\[z_1z_2 = r_1r_2e^{i(\theta_1 + \theta_2)}\]
\[
\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}
\]

If \(z = re^{i\theta}\), then \(\overline{z} = r e^{-i\theta}\) (Do you see why?) and so \(z\overline{z} = (re^{i\theta})(re^{-i\theta}) = r^2\).

**Example**

To calculate \((1 + i)^8\), we can first rewrite \(1 + i\) as \(\sqrt{2}e^{i\pi/4}\). Then
\[
\left(\sqrt{2}e^{i\pi/4}\right)^8 = (\sqrt{2})^8 e^{i8\pi/4} = 16e^{2\pi i} = 16.
\]

**Roots of Unity**

The equation
\[
z^n = 1
\]
has \(n\) complex-valued solutions, called the \(n^{th}\) roots of unity. Since we know each root has magnitude 1, let \(z = e^{i\theta}\). Then

\[
(e^{i\theta})^n = 1 \quad e^{in\theta} = e^{i(2\pi k)}
\]

\[
n\theta = 2\pi k \quad \theta = \frac{2\pi k}{n}
\]

so the \(n^{th}\) roots of unity are of the form
\[
z = e^{i\frac{2\pi k}{n}}.
\]

There are \(n\) distinct roots, after which we start duplicating roots already found.

\[
\sqrt{1^2 + 1^2} = \sqrt{2} \quad \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}
\]

\[
(e^{i\theta})^n = e^{in\theta} \quad \text{together with Euler's Equation, gives us deMoivre's Formula:}
\]

\[
(cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta
\]

\[
1 = e^{0i} = e^{2\pi ki}
\]

for \(k = 0, \pm 1, \pm 2, \ldots\)

These are evenly spaced around the unit circle.
Example
The 3rd roots of unity are

\[
e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}
\]
\[
e^{-i\frac{2\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}
\]

You can verify that \((-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1\) and \((-\frac{1}{2} - i\frac{\sqrt{3}}{2})^3 = 1\).

This tutorial has reviewed the basics of complex arithmetic. The methods of complex analysis, which build on this background, are both intriguing and powerful!

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**Key Concept**

<table>
<thead>
<tr>
<th>Standard Form</th>
<th>Polar Form</th>
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<tbody>
<tr>
<td>(z = a + bi)</td>
<td>(z = re^{i\theta})</td>
</tr>
<tr>
<td>(a = \text{Re } z)</td>
<td>(a = r \cos \theta)</td>
</tr>
<tr>
<td>(b = \text{Im } z)</td>
<td>(b = r \sin \theta)</td>
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<tr>
<td>(</td>
<td>z</td>
</tr>
<tr>
<td>(\overline{z} = a - bi)</td>
<td>(\theta = \text{arg } z)</td>
</tr>
<tr>
<td>(r = \sqrt{a^2 + b^2})</td>
<td>(z = re^{-i\theta})</td>
</tr>
<tr>
<td>(\tan \theta = \frac{b}{a})</td>
<td>(\overline{z} = re^{-i\theta})</td>
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</tbody>
</table>

Euler’s Equation,

\[e^{i\theta} = \cos \theta + i \sin \theta,\]

provides the connection between these two representations of complex numbers.

[I’m ready to take the quiz.] [I need to review more.]
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