Harvey Mudd College Math Tutorial:

Convergence Tests for Infinite Series

In this tutorial, we review some of the most common tests for the convergence of an **infinite series**

$$\sum_{k=0}^{\infty} a_k = a_0 + a_1 + a_2 + \cdots$$

The proofs or these tests are interesting, so we urge you to look them up in your calculus text.

Let

$$\begin{aligned} s_0 &= a_0 \\ s_1 &= a_1 \\ &\vdots \\ s_n &= \sum_{k=0}^{n} a_k \\ &\vdots \end{aligned}$$

If the sequence $\{s_n\}$ of **partial sums** converges to a limit L, then the series is said to **converge** to the **sum** L and we write

$$\sum_{k=0}^{\infty} a_k = L.$$

For $j \geq 0$, $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=j}^{\infty} a_k$ converges, so in discussing convergence we often just write $\sum a_k$.

Example

Consider the **geometric series**

$$\sum_{k=0}^{\infty} x^k.$$

The n^{th} partial sum is

$$s_n = 1 + x + x^2 + \cdots + x^n.$$

Multiplying both sides by x,

$$xs_n = x + x^2 + x^3 + \cdots + x^{n+1}.$$

Subtracting the second equation from the first,

$$(1 - x)s_n = 1 - x^{n+1},$$

so for $x \neq 1$,

$$s_n = \frac{1 - x^{n+1}}{1 - x}.$$

For $|x| < 1$,

$$\lim_{n\to\infty} s_n = \frac{1}{1-x}.$$

It is easy to see that $\sum\limits_{k=0}^{\infty} x^k$ diverges for $|x| \geq 1$. Thus $\sum\limits_{k=0}^{\infty} x^k = \frac{1}{1-x}$ for $|x| < 1$ and diverges for $|x| \geq 1$.

Divergence Test

If $\lim\limits_{k\to\infty} a_k \neq 0$, then $\sum\limits_{k=0}^{\infty} a_k$ diverges.

Example
The series $\sum\limits_{k=0}^{\infty} \frac{k}{2k+1}$ diverges, since $\lim\limits_{k\to\infty} \frac{k}{2k+1} = 1/2 \neq 0$.

Integral Test

Let $f(x)$ be continuous, decreasing, and positive for $x \geq 1$. Then $\sum\limits_{k=1}^{\infty} f(k)$ converges if and only if $\int\limits_1^{\infty} f(x)dx$ converges.

Example
Consider the **p-series**

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

Since

$$\int_1^{\infty} \frac{1}{x^p} dx = \begin{cases} \frac{1}{1-p} x^{1-p}\Big|_1^{\infty}, & p > 1 \\ \ln|x|\big|_1^{\infty}, & p = 1 \\ \frac{1}{1-p} x^{1-p}\Big|_1^{\infty}, & 0 < p < 1 \end{cases} = \begin{cases} \frac{1}{1-p} \\ \infty \\ \infty, \end{cases}$$

the series converges for $p > 1$ and diverges for $0 < p \leq 1$.

The divergent p-series

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

with $p = 1$ is called the **Harmonic Series.**

Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with non-negative terms. If $a_k \leq b_k$ for all k sufficiently large, then

1. If $\sum b_k$ converges, then $\sum a_k$ also converges.
2. If $\sum a_k$ diverges, then $\sum b_k$ also diverges.

Informally, if the "larger" series converges, so does the "smaller." If the "smaller" series divers, so does the "larger."

Examples

- Since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, so does $\sum_{k=1}^{\infty} \frac{1}{k^2+3}$. $\boxed{\frac{1}{k^2+3} < \frac{1}{k^2} \text{ for all } k.}$
- Since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges, so does $\sum_{k=1}^{\infty} \frac{1}{\ln|k+1|}$. $\boxed{\frac{1}{\ln|k+1|} > \frac{1}{k} \text{ for } k \geq 2.}$

Limit Comparison Test

Let $\sum a_k$ and $\sum b_k$ be series with positive terms. If

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L$$

where $0 < L < \infty$ then $\sum a_k$ and $\sum b_k$ either both converge or both diverge.

Example

The series $\sum_{k=1}^{\infty} \frac{k^2-1}{5k^3}$ diverges, since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges and

$$\lim_{k\to\infty} \frac{\frac{k^2-1}{5k^3}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k^2-1}{5k^2} = \frac{1}{5}.$$

Ratio Test

Let $\sum a_k$ be a series with positive terms and suppose that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = L.$$

1. If $L < 1$, then $\sum a_k$ converges.
2. If $L > 1$, then $\sum a_k$ diverges.
3. If $L = 1$, then the test is inconclusive.

Example

The series $\sum_{k=1}^{\infty} \frac{1}{k!}$ converges, since

$$\lim_{k \to \infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k \to \infty} \frac{1}{k+1} = 0.$$

Root Test

Let $\sum a_k$ be a series with non-negative terms and suppose that

$$\lim_{k \to \infty} (a_k)^{\frac{1}{k}} = L.$$

1. If $L < 1$, then $\sum a_k$ converges.
2. If $L > 1$, then $\sum a_k$ diverges.
3. If $L = 1$, then the test is inconclusive.

Example

The series $\sum_{k=0}^{\infty} \left(\frac{k}{2k+1}\right)^k$ converges, since

$$\lim_{k \to \infty} \left[\left(\frac{k}{2k+1}\right)^k\right]^{\frac{1}{k}} = \lim_{k \to \infty} \frac{k}{2k+1} = \frac{1}{2}.$$

Alternating Series Test

Consider the **alternating series**

$$\sum_{k=0}^{\infty} (-1)^k a_k$$

where $a_k > 0$ for all $k \geq 0$.

If $a_{k+1} < a_k$ for all k and $\lim a_k = 0$, then $\sum_{k=0}^{\infty} (-1)^k a_k$ converges.

Example

The series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$ converges, since $\frac{1}{(k+1)+1} < \frac{1}{k+1}$ and $\lim_{k \to \infty} \frac{1}{k+1} = 0$. This series is **conditionally convergent**, rather than **absolutely convergent**, since $\sum_{k=0}^{\infty} \left| \frac{(-1)^k}{k+1} \right| = \sum_{k=0}^{\infty} \frac{1}{k+1}$ diverges.

Key Concepts

The infinite series

$$\sum_{k=0}^{\infty} a_k$$

converges if the sequence of partial sums converges and diverges otherwise.

For a particular series, one or more of the common convergence tests may be most convenient to apply.

[I'm ready to take the quiz.] [I need to review more.]
[Take me back to the Tutorial Page]