Fundamental Theorem of Calculus

Let $f$ be continuous on $[a, b]$. If $F$ is any antiderivative for $f$ on $[a, b]$, then

$$\int_a^b f(t) \, dt = F(b) - F(a).$$

Here’s a sketch of the proof, based on Salas and Hille’s *Calculus: One Variable*.

Let $G(x) = \int_a^x f(t) \, dt$.

Then it may be proven that $G(x)$ is an antiderivative for $f$ on $[a, b]$. Let $F(x)$ be another antiderivative for $f$ on $[a, b]$. Then $G(x)$ and $F(x)$ are continuous on $[a, b]$ and satisfy $G'(x) = F'(x) = f(x)$ for all $x$ in $[a, b]$. It may be shown that $F(x)$ and $G(x)$ differ only by a constant:

$$G(x) = F(x) + C \quad \text{for some } C \text{ and all } x \in [a, b].$$

Now

$$G(a) = \int_a^a f(t) \, dt = 0,$$

so $0 = G(a) = F(a) + C$. Then $C = -F(a)$, so

$$G(x) = F(x) - F(a).$$

Letting $x = b$,

$$G(b) = F(b) - F(a),$$

so

$$\int_a^b f(t) \, dt = F(b) - F(a).$$
Notation

We often write $\int_a^b f(t) \, dt = F(t)|^b_a$ or $\int_a^b f(t) \, dt = F(t)|_{t=a}^{t=b}$ to emphasize the variable with respect to which we are integrating.

Example

$$\int_1^3 x \, dx = \frac{x^2}{2} \bigg|_1^3 = \frac{3^2}{2} - \frac{1^2}{2} = 4.$$ 

If we had chosen a different antiderivative $\frac{x^2}{2} + C$, the outcome would have been identical:

$$\int_1^3 x \, dx = \left(\frac{x^2}{2} + C\right) \bigg|_1^3 = \left(\frac{9}{2} + C\right) - \left(\frac{1}{2} + C\right) = \frac{9}{2} + C - \frac{1}{2} - C = 4.$$ 

Properties

- $\int_a^a f(x) \, dx = 0$.
- Interchanging the limits of integration
  $$\int_b^a f(x) \, dx = -\int_a^b f(x) \, dx.$$
- Linearity
  $$\int_a^b [\alpha f(x) + \beta g(x)] \, dx = \alpha \int_a^b f(x) \, dx + \beta \int_a^b g(x) \, dx.$$
Key Concepts

Fundamental Theorem of Calculus

Let $f$ be continuous on $[a,b]$. If $F$ is any antiderivative for $f$ on $[a,b]$, then

\[ \int_a^b f(t) \, dt = F(b) - F(a). \]