Harvey Mudd College Math Tutorial:

Fundamental Theorem of Calculus

We are all used to evaluating definite integrals without giving the reason for the procedure much thought. The definite integral is defined not by our regular procedure but rather as a limit of **Riemann sums**. We often view the definite integral of a function as the area under the graph of the function between two limits. It is not intuitively clear, then, why we proceed as we do in computing definite integrals. *The Fundamental Theorem of Calculus justifies our procedure of evaluating an antiderivative at the upper and lower limits of integration and taking the difference.*

Fundamental Theorem of Calculus

Let f be continuous on $[a, b]$. If F is any antiderivative for f on $[a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Here's a sketch of the proof, based on Salas and Hille's *Calculus: One Variable*.

Let $G(x) = \int_a^x f(t)\,dt$.

Then it may be proven that $G(x)$ is an **antiderivative** for f on $[a, b]$. Let $F(x)$ be another antiderivative for f on $[a, b]$. Then $G(x)$ and $F(x)$ are continuous on $[a, b]$ and satisfy $G'(x) = F'(x) = f(x)$ for all x in $[a, b]$. It may be shown that $F(x)$ and $G(x)$ differ only by a constant:

$$G(x) = F(x) + C \quad \text{for some } C \text{ and all } x \in [a, b].$$

Now

$$G(a) = \int_a^a f(t)\,dt = 0,$$

so $0 = G(a) = F(a) + C$. Then $C = -F(a)$, so

$$G(x) = F(x) - F(a).$$

Letting $x = b$,

$$G(b) = F(b) - F(a),$$

so

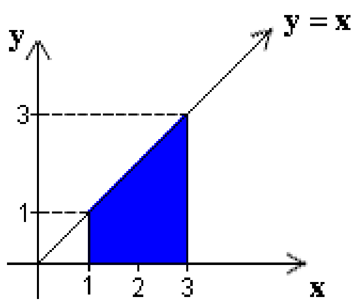
$$\int_a^b f(t)\,dt = F(b) - F(a).$$

Notation

We often write $\int_a^b f(t)\,dt = F(t)|_a^b$ or $\int_a^b f(t)\,dt = F(t)|_{t=a}^{t=b}$ to emphasize the variable with respect to which we are integrating.

Example

$$\begin{aligned}\int_1^3 x\,dx &= \left.\frac{x^2}{2}\right|_1^3 \\ &= \frac{3^2}{2} - \frac{1^2}{2} \\ &= 4.\end{aligned}$$



$Area_{shaded} = Area_{large\triangle} - Area_{small\triangle} = \frac{1}{2}(3^2) - \frac{1}{2}(1^2) = 4$

If we had chosen a different antiderivative $\frac{x^2}{2} + C$, the outcome would have been identical:

$$\begin{aligned}\int_1^3 x\,dx = \left.\left(\frac{x^2}{2} + C\right)\right|_1^3 &= \left(\frac{9}{2} + C\right) - \left(\frac{1}{2} + C\right) \\ &= \frac{9}{2} + C - \frac{1}{2} - C \\ &= 4.\end{aligned}$$

Properties

- $\int_a^a f(x)\,dx = 0.$
- **Interchanging the limits of integration**

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

- **Linearity**

$$\int_a^b [\alpha f(x) + \beta g(x)]\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Key Concepts

Fundamental Theorem of Calculus

Let f be continuous on $[a, b]$. If F is any antiderivative for f on $[a, b]$, then

$$\int_a^b f(t)\,dt = F(b) - F(a).$$

[I'm ready to take the quiz.] [I need to review more.]
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